PSet-01, Due on 19/08/2023, 11PM

[42 points], Kindly put in my mailbox near FB-382

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1. Argue Kepler's law from dimensional analysis starting from $F \propto 1/r^2$. What kind of scales are being ignored? [2 pts]

2. Prove that Fourier transform of $K^{-1}(r - r')$ is given by $1/\tilde{K}(k)$ where $\tilde{K}(k)$ is the Fourier transform of $K(r - r')$. [1 pt]

3. List all the typos that you spot in https://www-thphys.physics.ox.ac.uk/people/JohnCardy/rgft.pdf [4 pts]

4. You have seen that the tree level RG eigenvalue of the ϕ^4 coupling is $4 - \mathcal{D}$ in $\mathcal{D}$ dimensions. Find the tree level RG eigenvalue of the coupling corresponding to a term of the form $\nabla^m \phi^n$. [2 pts]

5. Perturbative RG

   (a) Consider
   $$Z = \text{Tr}\ e^{-H_* - \sum_i g_i \sum_r a^{\Delta_i} \phi_i(r)}$$
   around the fixed point. $\phi_i(r)$ are scaling operators of dimension $(\text{length})^{-\Delta_i}$. What is the dimension of g_i ? [1 pt]

   (b) Expand Z in powers of g_i. Show that the linear order term can be expressed as sum (both over i as well as lattice sites r) of one point functions. Write the expansion till $O(g_i g_j g_k)$. Express in continuum limit, i.e. replace the lattice sum $\sum_r$ appropriately. Note that the continuum integrals have in-built in them the lattice cut-off, points cannot come closer than a. [3 pts]

   (c) Note, that we are expanding analytically in g_i of a function around a point where it is supposed to be non-analytic in g_i [since this is a critical point]. What is the resolution to this puzzle? [2 pts]

   (d) Now change cut-off : $a \to a + \delta l$ and let us ask how g_i need to change to preserve Z. First identify where a dependence comes in both explicitly as well as implicitly.

      i. Show that the explicit dependence can be compensated via $g_i \to g_i + (\mathcal{D} - \Delta_i) g_i \delta l$ in the infinitesimal δl limit. Next the implicit dependence which enters through the continuum integrals :
      $$\int_{|r_1 - r_2| > a(1+\delta l)}$$
      can be made to look like the older cut-off by doing the split :
      $$\int_{|r_1 - r_2| > a(1+\delta l)} = \int_{|r_1 - r_2| > a} - \int_{a(1+\delta l) > |r_1 - r_2| > a}$$

. The first term gives back original contribution as the second order term in $O(g_i g_j)$, for the shell integral term, find how this can be re-absorbed by change of coupling into the one-point function term. Hint: Use

$$\phi_i(r_1)\phi_j(r_2) = \sum_k c_{ijk}\phi_k(\frac{r_1 + r_2}{2}).$$

and split integrals $d^{\mathcal{D}}r_1 d^{\mathcal{D}}r_2$ into sum and difference variables and do integral over the difference. [3 pts]

ii. Plugging everything together derive:

$$dg_k/dl = (\mathcal{D} - \Delta_k)g_k - \sum_{ij} c_{ijk}g_i g_j.$$

[2 pts]

6. Prove that the density of states in momentum space for $\Delta = 2$ can be expressed as convolution of two δ functions. [2 pt]

7. Special conformal transformations (SCT):

   (a) Derive the constrained form for $c_{\mu\nu\rho}$ which parametrizes the quadratic part of ϵ_μ. What degree of freedom is still left? [2 pt]

   (b) From here derive $\delta x^\mu = 2(x \cdot b)x^\mu - b^\mu x^2$. [2 pt]

   (c) Under the SCT show that the metric indeed changes by a conformal factor $\Lambda(x) = (1 - 2b \cdot x + b^2 x^2)^2$. [2 pt]

   (d) From the definition of generator : $iG^a\phi = \frac{\delta x^a}{\delta \epsilon^a}\partial_\mu\phi - \frac{\delta R}{\delta \epsilon^a}$ assuming that field does not change, derive : $K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2\partial_\mu)$. [3 pt]

   (e) Derive the relationship of $|x'_i - x'_j|$ in terms of $|x_i - x_j|$ under SCT. [3 pt]

   (f) Starting from assuming the action of conformal generators at $x = 0$ find action $K_\mu\phi(x)$. [4 pt]

   (g) Verify the commutator : $[K_\mu, P_\nu]$ and $[K_\mu, L_{\rho\sigma}]$. [2+2 pt]