# Introduction to $\mathcal{N} = 4$ Super Yang-Mills

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The supersymmetry algebra and its representation theory is introduced. We construct Lagrangian theories which are simultaneously invariant under supergauge transformations and under Yang-Mills transformations. Finally, a few properties of the $\mathcal{N} = 4$ Super Yang-Mills theory, such as its spectrum, conformal invariance and its importance in the AdS-CFT correspondence are summarised.

## INTRODUCTION

The familiar conserved quantities, such as energy-momentum, angular momentum, and charge, transform as vectors, tensors, and scalars under the Lorentz group. It is also possible for a conserved quantity to transform as a spinor. The Haag-Lopuszanki-Sohnius theorem[1] states that such fermionic symmetry generators can only belong to the (0,1/2) and (1/2,0) representations and that the *supersymmetry* algebra generated by them is the only symmetry of the S-matrix consistent with relativistic quantum field theory. In this report, we shall discuss this algebra and its consequences.

We first present the supersymmetry algebra:

$$\begin{aligned} \{Q_{\alpha r}, \bar{Q}_{\dot{\beta} s}\} &= 2\sigma^{\mu}_{\alpha\dot{\beta}} P_\mu \delta_{rs} \\ \{Q_{\alpha r}, Q_{\beta s}\} &= \{\bar{Q}_{\dot{\alpha} r}, \bar{Q}_{\dot{\beta} s}\} = 0 \\ [P_\mu, Q_{\alpha r}] &= [P_\mu, \bar{Q}_{\dot{\alpha} r}] = 0 \end{aligned} \tag{1}$$

The indices $(\alpha, \beta, \ldots, \dot{\alpha}, \dot{\beta}, \ldots)$ run from one to two and denote two-component Weyl spinors. Those with dotted indices transform under the $(0, 1/2)$ representation of the Lorentz group, while those with undotted indices transform under the $(1/2, 0)$ conjugate representation. The indices $r, s, \ldots$ distinguish different generators; they run from 1 to some number $N \geq 1$. The algebra with $N = 1$ is called the supersymmetry algebra, while those with $N > 1$ are called extended supersymmetry algebras.

## REPRESENTATIONS OF THE SUPERSYMMETRY ALGEBRA

The energy-momentum four-vector $P_\mu$ commutes with the supersymmetry generators $Q_{\alpha r}$ and $\bar{Q}_{\dot{\alpha} r}$. The mass operator $P^2$ is a Casimir operator, so irreducible representations of the supersymmetry algebra are of equal mass. We shall construct these irreducible representations considering fixed timelike and null momenta. It can be proved that every such representation contains an equal number of bosonic and fermionic states (see Appendix A).

**Massive states** Consider a massive, one-particle state with $P^2 = -M^2$. Boost to the rest frame, where $P_\mu = (-M, 0, 0, 0)$. (1) becomes

$$\begin{aligned} \{Q_{\alpha r}, \bar{Q}_{\dot{\beta} s}\} &= 2M\delta_{\alpha\dot{\beta}}\delta_{rs} \\ \{Q_{\alpha r}, Q_{\beta s}\} &= \{\bar{Q}_{\dot{\alpha} r}, \bar{Q}_{\dot{\beta} s}\} = 0. \end{aligned} \tag{2}$$

Rescale the generators

$$\begin{aligned} a_{\alpha r} &= \frac{1}{\sqrt{2M}} Q_{\alpha r} \\ a^+_{\alpha r} &= \frac{1}{\sqrt{2M}} \bar{Q}_{\dot{\alpha} r} \end{aligned} \tag{3}$$

to show that (2) is isomorphic to the algebra of $2N$ fermionic creation and annihilation operators:

$$\begin{aligned} \{a_{\alpha r}, a^+_{\dot{\beta} s}\} &= \delta_{\alpha\dot{\beta}}\delta_{rs} \\ \{a_{\alpha r}, a_{\beta s}\} &= \{a^+_{\dot{\alpha} r}, a^+_{\dot{\beta} s}\} = 0. \end{aligned} \tag{4}$$

The representations of this algebra are well known. They are constructed from a 'vacuum' $\Omega$, defined by the condition

$$a_{\alpha r}\Omega = 0.$$

The states are built by applying the creation operators:

$$\Omega^{(n)\alpha_1\ldots\alpha_n}_{r_1\ldots r_n} = \frac{1}{\sqrt{n!}} a^+_{\alpha_1 r_1} \ldots a^+_{\alpha_n r_n} \Omega.$$

Because the $a^+_{\alpha r}$ anticommute, $\Omega^n$ is antisymmetric under the exchange of two pairs of indices $\alpha_i r_i, \alpha_j r_j$. Each pair of indices takes 2N different values, so $n \leq 2N$. For any given n, there are $\binom{2N}{n}$ different states. Summing over all n, gives the dimension of the representation to be $2^{2N}$. The state with the highest spin is obtained by symmetrizing in as many spinor indices as possible. Because we simultaneously antisymmetrize in the second index, we may only symmetrize in N spinor indices. This leads to spin-$\frac{1}{2}N$. The highest spin in the above irreducible massive multiplet is then $j + \frac{1}{2}N$, where j is the spin of $\Omega$; it occurs exactly once.

**Massless states** We shall now analyze the massless case, $P^2 = 0$. Boost to the frame where $P_\mu = (-E, 0, 0, E)$. (1) becomes

$$\begin{aligned} \{Q_{\alpha r}, \bar{Q}_{\dot{\beta} s}\} &= 2\begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix}\delta_{rs} \\ \{Q_{\alpha r}, Q_{\beta s}\} &= \{\bar{Q}_{\dot{\alpha} r}, \bar{Q}_{\dot{\beta} s}\} = 0. \end{aligned} \tag{5}$$

The operators $Q_{2r}$ and $\bar{Q}_{\dot{2}r}$ are totally anticommuting and must therefore be represented by zero. Rescale the generators

$$\begin{aligned} a_r &= \frac{1}{2\sqrt{E}} Q_{1r} \\ a_r^+ &= \frac{1}{2\sqrt{E}} \bar{Q}_{\dot{1}r} \end{aligned} \tag{6}$$

to show that (2) is isomorphic to the algebra of $N$ fermionic creation and annihilation operators:

$$\begin{aligned} \{a_r, a_s^+\} &= \delta_{rs} \\ \{a_r, a_s\} = \{a_r^+, a_s^+\} &= 0. \end{aligned} \tag{7}$$

As before, the operators $a_r$ and $a_r^+$ raise and lower the helicity of a state by $\frac{1}{2}$ and we have

$$a_r \Omega_{\underline{\lambda}} = 0$$

for the state of lowest helicity, say, $\underline{\lambda}$. The states are built by applying the creation operators on the vacuum $\Omega_{\underline{\lambda}}$:

$$\Omega^{(n)}_{\underline{\lambda}+\frac{1}{2}n, r_1 \dots r_n} = \frac{1}{\sqrt{n!}} a^+_{r_1} \dots a^+_{r_n} \Omega_{\underline{\lambda}}. \tag{8}$$

This state has helicity $\underline{\lambda} + \frac{1}{2}n$, is antisymmetric in $r_1, \dots, r_n$, and is $\binom{N}{n}$-times degenerate. The state with highest helicity in this representation has helicity $\bar{\lambda} = \underline{\lambda} + \frac{1}{2}N$, The representation has dimension $2^N$, so we see that one massive representation splits into $2^N$ massless representations. In CPT-invariant theories, the number of states must be doubled, for CPT reverses the sign of the helicity; except if the multiplet is already CPT complete.

We showed that the dimension for massless representations is $2^N$. A corollary of a theorem of Weinberg and Witten[2] is : *A quantum field theory without gravity cannot contain massless states with helicity* $|\lambda| > 1$. $\bar{\lambda} = \underline{\lambda} + \frac{1}{2}N$ then shows that we must have $N \leq 4$.

Note that the SUSY algebra (1) is invariant under a group $U(N)$ of internal symmetries

$$Q_{\alpha r} \rightarrow \sum_s V_{rs} Q_{as},$$

with $V_{rs}$ an $N \times N$ unitary matrix. This is known as *R-symmetry*. It may or may not be a good symmetry of the action. (For $N = 4$, we will see that the diagonal $U(1)$ is broken, and only $SU(4)$ acts as an internal symmetry.) For $N = 1$, the conserved bosonic charge is called $R$:

$$[Q_\alpha, R] = Q_\alpha, [\bar{Q}_{\dot{\alpha}}, R] = -\bar{Q}_{\dot{\alpha}}.$$

## $\mathcal{N} = 4$ SPECTRUM

As told above, the $N = 4$ case is maximal. Clearly there is only one possible supermultiplet, having $\underline{\lambda} = -1$, $\bar{\lambda} = 1$. From (8) we see that this supermultiplet contains 1 boson of each helicity $\pm 1$, 4 fermions of each helicity $\pm 1/2$, and 6 bosons of helicity 0. This is a gauge multiplet because it contains helicity-1, and we conclude that the only interaction can be that of a non-Abelian gauge field $A_\mu$, in interaction with itself and its superpartners in the multiplet, the four fermions and six scalars, all in the adjoint representation of the gauge group.

We will restrict ourselves to $U(N)$ as the gauge group. As told above the *field* content is:

1. Gauge field $A_\mu : \mu = 0, \dots, 3$ is a Lorentz vector index.

2. Weyl spinors $\lambda^i_\alpha : i = 1, \dots, N = 4, \alpha$ is a spinor index.

3. Scalars $X^i : i = 1, \dots, 6$.

As the gauge field is necessarily in the adjoint representation of the gauge group $U(N)$, all of these fields are $N \times N$ matrices in the adjoint representation. The R-symmetry is $SU(4)$ (not $U(4)$; can be seen from the Lagrangian (9): any further $U(1)$ does not leave it invariant), it acts on the fermions $\lambda^i_\alpha$ in the fundamental representation of $SU(4)$. Indeed $SU(4) \sim SO(6)$ and the fundamental representation of $SU(4)$ is the spinor representation of $SO(6)$. From (8), the $X$'s are obtained by the action of two antisymmetrized $Q$'s on the ($SU(4)$ singlet) $A_\mu$, we expect it to transform in the antisymmetric tensor representation of $SU(4)$. This works out to be the standard vector 6 of $SO(6)$, which acts on the $i$ index on $X^i$.

The Lagrangian density (which is completely determined by SUSY, up to the parameter $g$) is

$$\begin{aligned} \mathcal{L} = \frac{1}{g^2}(Tr[F^2 + (DX^i)^2 + i\bar{\lambda}\not{D}\lambda \\ - \sum_{i<j}^{6} [X^i, X^j]^2. \end{aligned} \tag{9}$$

The *superpotential* $V = \frac{1}{g^2} Tr \sum_{i<j}^6 [X^i, X^j]^2$ is non-negative.

Suppressing the indices and Pauli matrices, the SUSY transformation laws essentialy are:

$$\begin{aligned} [Q, X] &= \lambda \\ \{Q, \lambda\} &= F^+ + [X, X] \\ \{Q, \bar{\lambda}\} &= DX \\ [Q, A] &= \lambda \end{aligned} \tag{10}$$

(9) is the most general renormalizable Lagrangian density consistent with $N = 4$ supersymmetry.

## CONFORMAL INVARIANCE OF $\mathcal{N}=4$ SYM

The beta function for $N=4$ Yang-Mills (9) vanishes identically by cancellation between the gauge field and matter contributions. Consequently, the coupling constant is scale-independent and the theory is conformally invariant.[1]

We will show that the 1-loop $\beta$ function for $N=4$ SYM vanishes. For a gauge theory with $N_f$ Weyl fermions and $N_s$ complex scalars([4]),

$$\mu\frac{dg}{d\mu} = -\frac{b}{16\pi^2}g^3 + \mathcal{O}(g^5), \tag{11}$$

where

$$b = \frac{11}{6}T(adj) - \frac{1}{3}\sum_a T(r_a) - \frac{1}{6}\sum_n T(r_n) \tag{12}$$

$a$ runs over fermions, $n$ over scalars, and $T(r)$ is the Dynkin index of the representation $r$. In our case everything is in the adjoint, and we obtain

$$b = \frac{T(adj)}{6}(11 - 2N_f - N_s). \tag{13}$$

In our case $N_f = 4, N_s = 3$, giving $b = 0$.

A similar argument shows that the 3-loop contribution to the $\beta$-function also vanishes([5]). A non-perturbative proof can be found in [7].

Therefore $N=4$ SYM has the following symmetries:

1. *Conformal Symmetry*, forming the group $SO(2,4) \sim SU(2,2)$ is generated by translations $P_\mu$, Lorentz transformations $J_{\mu\nu}$, dilations $D$ and special conformal transformations $K_\mu$,
2. *R-symmetry*, forming the group $SO(6) \sim SU(4)$,
3. *Poincaré supersymmetries* generated by the supercharges $Q_{\alpha r}$ and their adjoints $\bar{Q}_{\dot{\alpha}r}, r = 1, \ldots, 4$,
4. *Conformal supersymmetries* generated by the supercharges $S_{\alpha r}$ and their adjoints $\bar{S}_{\dot{\alpha}r}$. The presence of these symmetries results from the fact that the Poincaré supersymmetries and the special conformal transformations $K_\mu$ do not commute.(Because $[Q,\bar{Q}] = P$ and $[P,K] \neq 0$.) Since both are symmetries, their commutator must also be a symmetry, and these are the $S$ generators. So instead of the 16 supercharges of $N=4$ SUSY, we have 32 fermionic symmetry generators. (See Appendix B)

[1] Classically for renormalizable relativistic field theories, scale invariance implies conformal invariance. Quantum effects alter both symmetries, but in the present case the theory is exactly scale invariant at the quantum level, and the conformal group $SO(2,4)$ is a fully quantum mechanical symmetry. See [3].

## CONCLUSION

The particle content of the $N=4$ supermultiplet reveals a difficulty in incorporating $N=4$ supersymmetry in realistic theories of particles at accesible energies. The helicity $+1/2$ fermions belong to the supermultiplet along with helicity $+1$ gauge bosons. Gauge bosons belong to the adjoint representation of the gauge group, so if the supersymmetry generators are invariant under the gauge group then the helicity $+1/2$ fermions must also belong to the adjoint representation, which is real. This is in conflict with the fact that the known quarks and leptons belong to a representation of $SU(3)\times SU(2)\times U(1)$ which is *chiral* i.e. complex, hence different from the representation furnished by their CPT-conjugates, the helicity -1/2 fermions.

Nevertheless, the maximal $N=4$ gauge theory has many interesting properties, like the conformal invariance demonstrated above, which also holds at the quantum level; thus providing a non-trivial example of a 4D CFT. Given the huge successes in understanding 2D CFTs, one might hope that at least some of the aspects allowing their treatment in $D=2$ might fruitfully reappear in $D=4$. One of the many intriguing features of 2D CFTs is that they are intimately connected to integrable $2+0D$ lattice models in statistical mechanics, or, equivalently, to $1+1D$ quantum spin chains. Thus it is hoped that integrability might also play a role in $N=4$ SYM([6]).

$N=4$ SYM also appears in the AdS-CFT correspondence([7]) which states that at low energy $N=4$ SYM is equivalent to $IIB$ string theory in $\mathrm{AdS}_5\times S_5$. In string theory, *open* strings(i.e. strings with a boundary) can start or end on $Dp$-branes, which are subspaces of the $10D$ spacetime with $p$ spatial dimensions. The physical world should then correspond to $D3$-branes. Open strings spectra contain massless vector fields. The open strings that end on $Dp$-branes then correspond to a gauge field propagating in $p+1$-dimensions. Strings have two ends. Each end can be labelled by which brane it ends on. This means that the string states are $N\times N$ matrices, where $N$ is the number of $D$-branes. The reason $N=4$ SYM appears in the AdS-CFT correspondence is because the low energy effective action for open strings on a stack of N $D3$-branes in type $IIB$ string theory turns out to be precisely (9).

To see why, imagine placing a single $D3$-brane in $R^{3,1}$. This breaks 10 (total) $-$ 4 (along the brane) = 6 translational symmetries, and so we expect to have 6 Goldstone bosons in the theory of the $D3$ branes– these are the $X^i$s! Similarly, the full 10D theory had 32 supercharges (from $N=2$ supergravity in 10D(equivalent to $N=8$ SUSY in 4D which has 32 charges)), but we only see 16 supercharges remaining in the $N=4$ theory. To understand where they went, consider the supercharges $Q$ :

$$\{Q,Q\} \sim \gamma^\mu P_\mu.$$

Those $Q$ whose anticommutators generate translations along the 6 broken translational symmetries are broken, and this leaves us with 16 unbroken supersymmetries. By "Goldstino's theorem", this breaking results in 16 massless fermions, which can be rearranged into the $4\lambda^i_\alpha s$. This is another way to understand the $N = 4$ spectrum.

**Acknowledgements** The argument for how $N = 4$ SYM spectrum can be obtained from string theory is from McGreevy's course notes [8].

## APPENDIX A

Define a fermion number operator $N_F$, such that $(-1)^{N_F}$ has eigenvalue +1 on bosonic states and -1 on fermionic states. We have

$$(-1)^{N_F} Q_{\alpha r} = -Q_{\alpha r}(-1)^{N_F} \tag{14}$$

Using (14) and the cyclic property of the trace, it follows that

$$Tr[(-1)^{N_F}\{Q_{\alpha r}, \bar{Q}_{\dot{\beta} s}\}] = 0 \tag{15}$$

From (1),

$$\begin{aligned} Tr[(-1)^{N_F}\{Q_{\alpha r}, \bar{Q}_{\dot{\beta} s}\}] = 2\sigma^{\mu}_{\alpha\dot{\beta}}\delta_{rs} Tr[(-1)^{N_F} P_\mu] \\ = 0. \end{aligned} \tag{16}$$

For fixed non-zero $P_\mu$, this reduces to

$$Tr(-1)^{N_F} = 0, \tag{17}$$

proving that supersymmetry representations contain equal number of bosonic and fermionic states.

## APPENDIX B

In this appendix, we shall explore the $N = 1$ superconformal algebra, and then generalise the result to $N > 1$.

**Notation** For any $N$, we can define Majorana 4-spinors:

$$Q_r = \begin{pmatrix} Q_{\alpha r} \\ \bar{Q}_{\dot{\alpha} r} \end{pmatrix} \quad ; \quad \bar{Q}_r = Q_r^\dagger \gamma^0 = \left(Q_{\alpha r}, \bar{Q}_{\dot{\alpha} r}\right). \tag{18}$$

The SUSY algebra becomes

$$\{Q_r, \bar{Q}_s\} = 2\delta_{rs}\gamma^\mu P_\mu \quad ; \quad [Q, R] = i\gamma_5 Q. \tag{19}$$

Usually, one introduces *anticommuting* spinor parameters $\xi^\alpha_r, \bar{\xi}_{\dot{\alpha} r}$. For a Majorana spinor $\xi_r$ derived from $\xi^\alpha_r$, we use the summation convention:

$$\xi^\alpha_r Q_{\alpha r} + \bar{\xi}_{\dot{\alpha} r}\bar{Q}^{\dot{\alpha}}_r = \bar{\xi} Q \quad ; \quad \bar{\xi} = \xi^\dagger \gamma^0. \tag{20}$$

For a field $\phi$ in a supermultiplet, we can then define infinitesimal supersymmetry variations:

$$\delta\phi = -i[\phi, \xi^\alpha_r Q_{\alpha r} + \bar{\xi}_{\dot{\alpha} r}\bar{Q}^{\dot{\alpha}}_r] = -i[\phi, \bar{\xi} Q]. \tag{21}$$

**Supercurrent** As for any continuous symmetry, there should be a Noether supercurrents $J_{\mu r}$ associated with the SUSY transformations (21):

$$Q_r = \int d^3x\, J_{0r}(x). \tag{22}$$

From (21),

$$\delta J_{\mu r} = -i[J_{\mu r}, \bar{\xi} Q], \tag{23}$$

we get by integration over $x$,

$$\int d^3x \delta J_{0r} = -i[Q_r, \bar{\xi} Q] = -2i\gamma^\mu \xi P_\mu. \tag{24}$$

The momentum four-vector, on the other hand, is the $x$ integral over components of the energy-momentum tensor (stress tensor):

$$P_\mu = \int d^3x\, T_{0\mu}. \tag{25}$$

We will assume that $J_{\mu r}$ are $\gamma-$traceless, and $T_{\mu\nu}$ is symmetric and traceless:

$$\begin{aligned} 0 = \partial^\mu J_{\mu r} = \partial^\mu T_{\mu\nu} \\ 0 = \gamma^\mu J_{\mu r} = T_{\mu\nu} - T_{\nu\mu} = T^\mu_\mu. \end{aligned} \tag{26}$$

The symmetry of $T_{\mu\nu}$ allows the construction of a further conserved current as a first moment of $T_{\mu\nu}$:

$$\begin{aligned} m_{\mu\nu\rho} = x_\mu T_{\nu\rho} - x_\nu T_{\mu\rho} \quad ; \quad \partial^\rho m_{\mu\nu\rho} = 0 \\ J_{\mu\nu} = \int d^3x\, m_{\mu\nu 0}. \end{aligned} \tag{27}$$

The tracelessness conditions (26) allow the construction of further conserved moments:

$$\begin{aligned} d_\mu = x^\nu T_{\mu\nu} \quad &; \quad \partial^\mu d_\mu = 0 \\ k_{\mu\rho} = x_\mu x^\nu T_{\nu\rho} - x^2 T_{\mu\rho} \quad &; \quad \partial^\rho k_{\mu\rho} = 0 \\ s_{\mu r} = i x^\nu \gamma_\nu J_{\mu r} \quad &; \quad \partial^\mu s_{\mu r} = 0 \\ &. \end{aligned} \tag{28}$$

The conserved charges are:

$$\begin{aligned} D &= \int d^3x\, d_0 \\ K_\mu &= \int d^3x\, k_{\mu 0} \\ S_r &= \int d^3x\, s_{0r} \\ &. \end{aligned} \tag{29}$$

(Note: $S$ is has a spinor index: (29) reads $S_\alpha = \int d^3x\, s_{0\alpha}$. This comes from the spinor index on $J_\mu$. For instance (27) is $0 = (\gamma^\mu)_{\alpha\beta} J_{\mu\beta}$.)

$N = 1$ **superconformal algebra** From (25),(27), assuming that the action of $P_\mu$ on fields is given by the derivative, and assuming surface integrals vanish, we obtain:

$$\begin{aligned}[P_\mu, P_\nu] &= \int d^3x\, [T_{0\mu}, P_\nu] = i \int d^3x\, \partial_\nu T_{0\mu} \\ &= 0 \\ [J_{\mu\nu}, P_\rho] &= i \int d^3x\, (x_\mu \partial_\rho T_{\nu 0} - x_\nu \partial_\rho T_{\mu 0}) \\ &= i(\eta_{\nu\rho} P_\mu - \eta_{\nu\rho} P_\mu) \\ &. \end{aligned} \tag{30}$$

Similarly we can show that:

$$\begin{aligned}[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho}) \\ [K_\mu, J_{\rho\sigma}] &= i(\eta_{\mu\rho} K_\sigma - \eta_{\mu\sigma} K_\rho)) \\ [D, J_{\mu\nu}] &= [K_\mu, K_\nu] = 0 \\ [P_\mu, D] &= i P_]mu \quad ; \quad [K_\mu, D] = -i K_\mu \\ [P_\mu, K_\nu] &= 2i(\eta_{\mu\nu} D - J_{\mu\nu}) \\ &. \end{aligned} \tag{31}$$

(30),(31) give the algebra of the *conformal group.* The relationship of supersymmetry with the conformal algebra is completely defined by the commutator $[Q, K_\mu]$. From (21),(28),(29):

$$\begin{aligned}[\bar{\xi} Q, K_\mu] &= -i \int d^3x\, (2 x_\mu x^\nu \delta T_{\nu 0} - x^2 \delta T_{\mu 0}) = \bar{\xi} \gamma_\mu S \\ &\implies [Q, K_\mu] = \gamma_\mu S. \end{aligned} \tag{32}$$

This is sufficient to determine all relationships of the algebra of *conformal supersymmetry*, once the SUSY algebra (19) and the conformal algebra (30),(31) are given:

$$\begin{aligned}[Q, J_{\mu\nu}] &= \frac{1}{2} \sigma_{\mu\nu} Q; \quad [S, J_{\mu\nu}] = \frac{1}{2} \sigma_{\mu\nu} S \\ [Q, D] &= \frac{1}{2} i Q; \quad [S, D] = -\frac{1}{2} i S \\ [Q, P_\mu] &= 0; \quad [S, P_\mu] = \gamma_\mu Q \\ [Q, K_\mu] &= \gamma_\mu S; \quad [S, K_\mu] = 0 \\ [Q, R] &= i \gamma_5 Q; \quad [S, R] = -i \gamma_5 S \\ [R, J_{\mu\nu}] &= [R, P_\mu] = [R, D] = [R, K_\mu] = 0 \\ \{Q, \bar{Q}\} &= 2\gamma^\mu P_\mu; \quad \{S, \bar{S}\} = 2\gamma^\mu K_\mu \\ \{S, \bar{Q}\} &= 2iD + \sigma^{\mu\nu} J_{\mu\nu} + 3i\gamma_5 R. \end{aligned} \tag{33}$$

We did not define a current for $R$, instead can use the last equation in (33) to define the charge $R$. Cannot set $R = 0$, since the commutator $[Q, R]$ is not zero.

$N > 1$ **superconformal algebra** Define 4-component *conformal spinors*:

$$\Sigma = \begin{pmatrix} Q_\alpha \\ \bar{S}_{\dot{\alpha}} \end{pmatrix}. \tag{34}$$

Also define additional 'Lorentz' generators:

$$J_{\mu 5} = \frac{1}{2}(P_\mu - K_\mu); \quad J_{\mu 6} = \frac{1}{2}(P_\mu + K_\mu); \quad J_{56} = -D. \tag{35}$$

Conformal algebra (30),(31) can then be written in a single line:

$$[J_{ab}, J_{cd}] = i(\eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}). \tag{36}$$

The conformal group is thus $SO(2,4)$. Also define:

$$\sigma_{\mu 5} = i\gamma_\mu \gamma_5; \quad \sigma_{\mu 6} = \gamma_\mu; \quad \sigma_{56} = \gamma_5. \tag{37}$$

$N = 1$ superconformal algebra (33) can then be written in three lines:

$$\begin{aligned}[\Sigma, J_{ab}] &= \frac{1}{2} \sigma_{ab} \Sigma; \; [\bar{\Sigma}, J_{ab}] = -\frac{1}{2} \sigma_{ab} \bar{\Sigma} \\ [\Sigma, R] &= \Sigma; \; [\bar{\Sigma}, R] = -\bar{\Sigma}; \; [J_{ab}, R] = 0 \\ \{\Sigma, \Sigma\} &= \{\bar{\Sigma}, \bar{\Sigma}\} = 0; \; \{\Sigma, \bar{\Sigma}\} = \sigma^{ab} J_{ab} - 3R. \end{aligned} \tag{38}$$

For $N > 1$, we have $\Sigma_r, r = 1, \ldots, N$. One possible extension of the algebra (35) is:

$$\begin{aligned}[\Sigma_r, J_{ab}] &= \frac{1}{2} \sigma_{ab} \Sigma_r; \; [\bar{\Sigma}_r, J_{ab}] = -\frac{1}{2} \sigma_{ab} \bar{\Sigma}_r \\ \{\Sigma_r, \Sigma_s\} &= \{\bar{\Sigma}_r, \bar{\Sigma}_s\} = 0; \; \{\Sigma_r, \bar{\Sigma}_s\} = \delta_{rs} \sigma^{ab} J_{ab} - 4R_{rs}. \end{aligned} \tag{39}$$

As before the last equation defines the charges $R_{rs}$. The Jacobi identities fix the remaining commutators. From Hermiticity property of the last equation,

$$R^\dagger_{rs} = R_{rs}. \tag{40}$$

Thus the traceless part of the $R_{rs}$ are $SU(N)$ generators.

**Superconformal Multiplets** (33) shows that $\{S_r, \bar{S}_s\} = 2\delta_{rs} \gamma^\mu K_\mu$, just like the $Q_r$. Then, as for $Q$'s, successive application of $S$ to any operator must at some point yield 0. Therefore one defines a *superconformal primary operator* $O$ such that:

$$[S, O]_\pm = 0, \quad O \neq 0, \tag{41}$$

$\pm$ if $O$ is bosonic and fermionic respectively. This is a stronger condition than annihilation by the $K_\mu$ as for *conformal primary operators.* An operator $O$ is called a *superconformal descendant operator* of an operator $O'$ if:

$$O = [Q, O']_\pm. \tag{42}$$

Clearly $O, O'$ belong to the same superconformal multiplet.

A superconformal primary operator cannot the $Q$-commutator of another operator. From (10), the superconformal primary operators of $N = 4$ SYM involve only the $X^i$.

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