# Random Matrix Behaviour in CFTs

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In this article, we try to identify signatures of chaos in 2D CFTs. The Gultzwiller trace formula gives us an identity followed by chaotic systems in the semi-classical limit. Here, we find an analogous identity for 2D CFTs which arise from the SL(2, $\mathbb{Z}$) symmetry of Virasoro primary density of states. We also draw an analogy from Berry's diagonal approximation by extracting coarse-grained spectral statistics of individual 2D CFTs and thus identifying signatures of chaos.

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## INTRODUCTION

The *random matrix theory* (RMT) suggests that the Hamiltonian of a chaotic quantum system, within an energy window where the density of states remains constant, looks like a random matrix in a generic basis. For systems that satisfy time-reversibility, the random elements of the matrix are going to follow the Gaussian orthogonal ensemble (GOE), whereas, for systems not satisfying time-reversibility, the elements will follow the Gaussian unitary ensemble (GUE). It was proposed that the elements of an operator following the RMT will have a form of,

$$O_{mn} = \bar{O}\delta_{mn} + \sqrt{\frac{\bar{O^2}}{D}} R_{mn}. \tag{1}$$

In Eqn. (1), $O_{mn}$ is the $mn$th element of the operator matrix. $\bar{O}$ is the ensemble average of $O$, and $R_{mn}$ is a random number following a particular distribution depending upon the system's symmetries.

*Gutzwiller trace formula [2]:* Gutzwiller 1971 introduced a trace formula for the dynamics of a classically chaotic system in its semiclassical limit. The Gutzwiller trace formula expresses the density of states of the system in the semiclassical limit as,

$$\rho(E) = \bar{\rho}(E) + \rho_{osc}(E). \tag{2}$$

In Eqn. (2), $\bar{\rho}(E)$ is the mean density, and $\rho_{osc}(E)$ is the oscillatory part given by a sum over the periodic orbits in the semiclassical limit $\gamma$ as,

$$\rho_{osc}(E) = \frac{1}{\pi} \mathrm{Re}\left(\sum_{\gamma} A_\gamma e^{iS_\gamma(E)}\right) \tag{3}$$

The oscillatory part encodes the information regarding the correlation between various energy levels of the system.

In the next section, we will explore the spectral decomposition of the SL(2, $\mathbb{Z}$) group. Using SL(2, $\mathbb{Z}$) spectral decomposition of torus partition functions of parity-invariant Virasoro CFTs, we show that spectral decomposition of the density of states is directly analogous to a trace formula.

## SPECTRAL DECOMPOSITION OF SL(2, $\mathbb{Z}$)

The SL(2, $\mathbb{Z}$) group comprises all transformations produced by a matrix of the form,

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $a, b, c, d \in \mathbb{Z}$, and $\det(M) = ad - bc = 1$.

A square-integrable function that is invariant under SL(2, $\mathbb{Z}$) transformation is given by [1] [3],

$$f(\tau) = \langle f \rangle + \int \{f, E_s\} E_s^*(\tau)\, ds + \sum_{n=1}^{\infty} (f, \phi_n)\phi_n(\tau). \tag{4}$$

In Eqn. (4), $\tau = x + iy$, $\langle f \rangle$ is the modular average of $f$. $E_s^*(\tau)$ is the completed Eisentein series, and $\phi_n(\tau)$ are the Maas cusps. $\{f, E_s\} = \frac{(f, E_s)}{\Lambda(s)}$, where, $(f, E_s)$

and $(f, \phi_n)$ are Peterson inner products with hyperbolic measure $dx\, dy/y^2$.

Considering a torus partition function $Z(\tau)$, the primary partition function, after removing the Virasoro symmetries, is given by,

$$Z_p(\tau) = \sqrt{y}|\eta(\tau)|^2 Z(\tau).$$

$Z_p(\tau)$ can be written in a modular invariant fashion as,

$$Z_p(\tau) = \hat{Z}_L(\tau) + Z_{spec}(\tau). \tag{5}$$

In Eqn. (5), $\hat{Z}_L(\tau)$ is the 'modular completion' of the 'light states'. This is first done by constructing a partition for light primaries defined as,

$$Z_L(\tau) = \sqrt{y} \sum_{\min(h,\bar{h})\leq\xi} q^{h-\xi}\bar{q}^{\bar{h}-\xi},$$

where, $q = e^{i2\pi\tau}$, and $\xi = \frac{c-1}{24}$. Then, we attain modular completion by taking the Poincare sum over all the SL(2, $\mathbb{Z}$) images of the light partition functions as,

$$\hat{Z}_L(\tau) = \sum_{\gamma\in SL(2,\mathbb{Z})} Z_L(\gamma\tau).$$

After explicitly constructing $\hat{Z}_L(\tau)$, we can obtain $Z_{spec}(\tau)$ from $Z_p(\tau)$ by subtracting. Now, $Z_{spec}(\tau)$ being square integrable, follows the spectral decomposition of SL(2, $\mathbb{Z}$) (Eqn. (4)) as,

$$Z_{spec}(\tau) = \langle Z_{spec}\rangle + \int \{Z_{spec}, E_s\} E_s^*(\tau)\, ds + \sum_{n=1}^{\infty} (Z_{spec}, \phi_n)\phi_n(\tau).$$

$Z_{spec}(\tau)$ is used to probe the chaotic high energy spectrum.

## TRACE FORMULAS FOR CHAOTIC SYSTEMS

### Semiclassical Systems

As mentioned, the Gutzwiller trace formula gives us a relation for the density of chaotic systems in the semiclassical limit, mentioned in Eq.. (2), as [2],

$$\rho(E) = \bar{\rho}(E) + \rho_{osc}(E).$$

The correlation between the various energy levels of the system can be extracted by micro-canonical coarse-graining the spectrum over an energy window $\delta E$. One important use of this trace formula is to find the coarse-grained micro-canonical spectral form factor, which is defined as,

$$K(E) = \int_{-\infty}^{\infty} e^{i\epsilon T} \overline{\rho_{osc}\left(E + \frac{\epsilon}{2}\right)\rho_{osc}\left(E - \frac{\epsilon}{2}\right)}\, d\epsilon, \tag{6}$$

where the overline denotes the average over an ensemble of statistically similar systems. Replacing Eqn. (3) in Eqn. (6), we get,

$$K(E) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} e^{i\epsilon T} \times \left(\sum_{\gamma_1,\gamma_2} \overline{e^{i\left(S_{\gamma_1}(E+\frac{\epsilon}{2}) - S_{\gamma_2}(E-\frac{\epsilon}{2})\right)}} + c.c.\right) d\epsilon.$$

The leading order contribution in level statistics comes from the periodic orbits for which $\Delta S = (S_{\gamma_1} - S_{\gamma_2}) = 0$. Berryu showed that when we restrict the summation to the orbits only when $\gamma_1 = \gamma_2$, at large times (when $T \to \infty$), we approximately get a linear ramp in the spectral form factor as,

$$K(E)|_{diagonal} \approx \frac{T}{2\pi} C_{RMT},$$

where $C_{RMT}$ is a constant depending on the type of the RMT distribution, and hence the Hamiltonian. $C_{RMT} = 2$ for GOE and $C_{RMT} = 1$ for GUE ensembles. This is known as Berry's diagonal approximation.

Berry's analysis takes the oscillatory behaviour of the density of states and uses it to extract, without ensemble averaging, the random matrix behaviour of a single chaotic quantum system. Despite the apparent factorisation of the original double sum over orbits, the diagonal approximation shows an emergent non-factorization by removing minute details from the spectrum. Now we ask if there is a similar relation between the density of states of a 2D CFT as in Eqn. (2).

### 2D CFTs

The partition function for primary density of states with spin-$j$ ($\rho_j(\Delta)$) in a CFT is defined as,

$$Z_p(\tau) = \sqrt{y}\sum_{j=0}^{\infty}(2-\delta_{j,0})\cos(2\pi j x)\int_j^{\infty} e^{-2\pi y(\Delta-2\xi)} \times \rho_j(\Delta)\, d\Delta \tag{7}$$

We first try transforming the spectral decomposition in Eqn. (4) into a microcanonical ensemble. The density equation can be written as,

$$\rho_j(t) = \hat{\rho}_{L,j}(t) + \rho_{spec,j}(t). \tag{8}$$

Here, $t$ is the "reduced twist" defined as,

$$t = \frac{\Delta - j}{2} - \xi.$$

Using inverse Laplace transform, the *spectral* part of the density matrix decomposition is given by [1],

$$\rho_{spec,j}(t) = \int \{Z_{spec}, E_{\frac{1}{2}+i\omega}\}\rho^*_{\frac{1}{2}+i\omega,j}(t) + \sum_{n=1}^{\infty} (Z_{spec}, \phi_n)\, \rho_{n,j}(t), \tag{9}$$

where, for $J \neq 0$,

$$\rho^*_{\frac{1}{2}+i\omega,j}(t) = a_j^{(s)}\theta(t)\frac{\cos\left(\omega \cosh^{-1}\left(\frac{2t}{j}+1\right)\right)}{\sqrt{t(t+j)}}$$

$$\rho_{n,j}(t) = b_j^{(s)}\theta(t)\frac{\cos\left(\omega_n \cosh^{-1}\left(\frac{2t}{j}+1\right)\right)}{\sqrt{t(t+j)}}.$$

Here, $a_j$'s and $b_j$'s are sporadic numbers with no definite structure. While, for $j = 0$, $b_0 = 0$, and,

$$\rho^*_{\frac{1}{2}+i\omega,j}(t) = \frac{\zeta(2i\omega}{t}(4t)^{i\omega}.$$

Now, we analyse these formulas. Both $\hat{\rho}_{L,j}(t)$ and $\rho_{spec,j}(t)$ are continuous in $t$. In the asymptotic spectrum, the mean level spacing of $\rho_j(t)$ goes approximately as $\exp(\sqrt{t})$. Thus, while there is an exponential growth of $\hat{\rho}_{L,j}(t)$ with $t$, we observe a highly oscillatory behaviour in $\rho_{spec,j}(t)$.

*The trace formula:* The smooth exponential growth of $\hat{\rho}_{L,j}(t)$ and the oscillatory behaviour of $\rho_{spec,j}(t)$ is analogous to those in Eq. (2). The connection can be drawn (for every value of $j$) as:

$$\bar{\rho}(E) \longleftrightarrow \hat{\rho}_{L,j}(t),$$

$$\rho_{osc}(E) \longleftrightarrow \rho_{spec,j}(t).$$

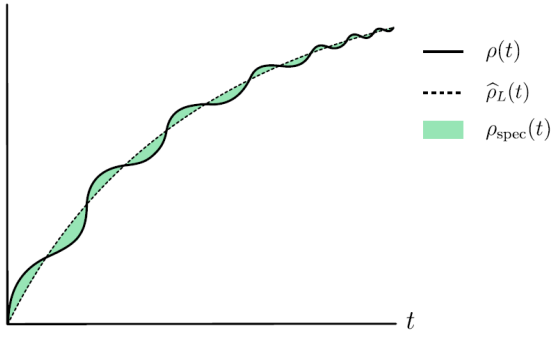


FIG. 1: Schematic figure showing the behaviour of $\rho(t)$ and those of $\hat{\rho}_L(t)$ and $\rho_{spec}(t)$ [1].

Now we write $A_\gamma$ as $A'_\gamma T_\gamma(E)$, where $A'_\gamma$ is the one-loop determinant over the orbit and $T_\gamma(E)$ is the time period of the $\gamma$th orbit. Thus,

$$T_\gamma(E) = \frac{\partial S_\gamma(E)}{\partial E}.$$

Now we make the following identifications,

$$S_{\omega,j} = \omega \cosh^{-1}\left(\frac{2t}{j}+1\right) + \omega\log(j) \tag{10}$$

$$T_{\omega,j} = \frac{\omega}{\sqrt{t(j+t}} \tag{11}$$

$$A'_{\omega,j} = \frac{\pi}{\omega}c_j^{(s)} j^{-i\omega}, \tag{12}$$

$c_j^{(s)}$ can be $a_j$'s or $b_j$'s depending on whether we are dealing with the Eisenstein terms or the Maas cusps.

Using these, we can write $\rho^*_{\frac{1}{2}+i\omega,j}(t)$ as,

$$\rho^*_{\frac{1}{2}+i\omega,j}(t) = \frac{1}{\pi}\,\mathrm{Re}\left(A'_{\omega,j}T_{\omega,j}e^{iS_{\omega,j}(t)}\right).$$

Thus, for any $j$, Eqn. (9) can be written as,

$$\rho_{spec,j}(t) = \frac{1}{\pi}\,\mathrm{Re}\left(\int \{Z_{spec}, E_{\frac{1}{2}+i\omega}\}A'_{\omega,j}T_{\omega,j}e^{iS_{\omega,j}(t)}\right) + \frac{1}{\pi}\,\mathrm{Re}\left(\sum_{n=1}^{\infty}(Z_{spec},\phi_n)\,A'_{\omega,j}T_{\omega,j}e^{iS_{\omega,j}(t)}\right). \tag{13}$$

We should note here that while the *Gutzwiller trace* formula is only valid for states at the high energy limit of the spectrum in the semiclassical limit, the 2D CFT trace, thus derived from the SL(2, $\mathbb{Z}$) spectral decomposition, is valid for all energies.

## BERRY DIAGONAL APPROXIMATION [1]

We now come to the last section of this article. Here, we try to find an analogous condition for *Berry's diagonal approximation* in 2D CFTs. We understand that averaging the oscillatory part over a large enough energy window renders the oscillatory contribution zero for chaotic systems in the semiclassical limit. Thus, $\overline{e^{iS_\gamma(E)}} = 0$, or, $\overline{\rho_{osc}(E)} = 0$. We draw an analogy and claim that the second term in Eqn. (8) is rendered zero when averaged over a large enough window $\delta t$ (which depends on the spectrum). we show this by averaging it over $(t-\delta t, t+\delta t)$. Thus,

$$\overline{\rho^*_{\frac{1}{2}+i\omega,j}(t)} = \frac{1}{\pi\delta t}\,\mathrm{Re}\left(A'_{\omega,j}\int_{t-\delta t}^{t+\delta t} T_{\omega,j}(t')e^{iS_{\omega,j}(t')}\,dt'\right) = 0. \quad (14)$$

This implies that $\overline{\rho_{spec,j}(t)} = 0$. Hence, $\overline{\rho_j(t)} = \rho^{\hat{}}_{L,j}(t)$. The behaviour of $\rho^{\hat{}}_{L,j}(t)$ is self-averaging, like in a microcanonical ensemble with an exponential argument rather than an oscillatory one.

Now, let us turn to the two-point functions. We define the two-point functions from the product of two densities and integrate them over a mean $t$ of $\frac{t_1+t_1}{2}$ and a difference $\epsilon = \frac{t_1-t_2}{2}$. Thus,

$$\overline{\rho^*_{\frac{1}{2}+i\omega_1,j_1}(t+\epsilon)\rho^*_{\frac{1}{2}+i\omega_2,j_2}(t-\epsilon)} = \frac{A_{\omega_1,j_1}A_{\omega_1,j_1}}{4\pi^2}T_{\omega_1,j_1}T_{\omega_1,j_1}\overline{e^{i\left(S_{\omega_1,j_1}(t+\epsilon)-S_{\omega_2,j_2}(t-\epsilon)\right)}} + c.c. \quad (15)$$

The terms with the sum of actions are zero because they are highly oscillatory. Instead, the terms with differences contribute more. The leading order contribution comes from the terms with $\Delta S = S_{\omega_1,j_1} - S_{\omega_2,j_2} = 0$. This is analogous to Berry's diagonal approximation.

### Appendix A: The Random Matrix Theory (RMT)

Wigner proposed that within an energy window in which the density of states of the system remains constant, then the Hamiltonian of a non-integrable system, on a generic basis, would look like a random matrix. Hence, the properties of a non-integrable system can be studied by studying the statistical properties of the random matrix [4].

### Wigner-Dyson level statistics

We take the example of a $2\times 2$ matrix where the elements have been randomly chosen from a Gaussian distribution,

$$H = \begin{bmatrix} \epsilon_1 & \frac{V}{\sqrt{2}} \\ \frac{V^*}{\sqrt{2}} & \epsilon_2 \end{bmatrix}.$$

The eigenvalues of the matrix are

$$E_{1,2} = \frac{\epsilon_1+\epsilon_2}{2} \pm \frac{1}{2}\sqrt{(\epsilon_1-\epsilon_2)^2+|V|^2}.$$

Defining $\omega = |E_1 - E_2|$, we can find the probability of having an energy difference of $\omega$ between two levels $(P(\omega))$. If we have time-reversal symmetry, the Hamiltonian would be symmetric and $V = V^*$. Thus,

$$P_1(\omega) = \frac{1}{(2\pi)^{3/2}\sigma^3}\iiint d\epsilon_1\, d\epsilon_2\, dV \exp\left(-\frac{\epsilon_1^2+\epsilon_2^2+V^2}{2\sigma^2}\right) \delta\left(\sqrt{(\epsilon_1-\epsilon_2)^2+|V|^2}-\omega\right).$$

We have chosen the matrix elements from a Gaussian ensemble with 0 mean and variance $\sigma^2$. This forms our Gaussian orthogonal ensemble (GOE).
In the case where $H \neq H^*$, the probability distribution will be given by,

$$P_2(\omega) = \frac{1}{(2\pi)^{3/2}\sigma^3}\iiiint d\epsilon_1\, d\epsilon_2\, d(Re(V))\, d(Im(V)) \exp\left(-\frac{\epsilon_1^2+\epsilon_2^2+|V|^2}{2\sigma^2}\right)\delta\left(\sqrt{(\epsilon_1-\epsilon_2)^2+|V|^2}-\omega\right),$$

where $|V|^2 = (Re(V))^2 + (Im(V))^2$. The extra integral comes from the freedom of choice of $V$'s real and imaginary parts of $V$ separately. This forms our Gaussian unitary ensemble (GUE).
Solving the integrals for GOE, we have,

$$P(\omega) = \frac{\omega}{2\sigma^2}\exp\left[-\frac{\omega^2}{4\sigma^2}\right], \quad (16)$$

and for GUE,

$$P(\omega) = \frac{\omega^2}{2\sqrt{\pi}(\sigma^2)^{3/2}}\exp\left[-\frac{\omega^2}{4\sigma^2}\right]. \quad (17)$$

In either of the cases, we see at $\omega \to 0$, $P(\omega) = 0$. In other words, as the level spacing vanishes, the probability of having two closely spaced energy states vanishes. This phenomenon is known as *level repulsion* and is a characteristic of a non-integrable system. Both Eq. (16) and Eq. (17) are Wigner-Dyson distributions.

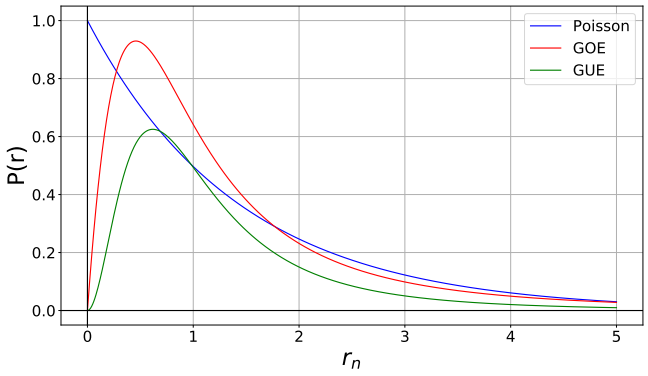


FIG. 2: $P(r)$ vs $r_n$) plots for different distributions. The *blue* line represents Poisson distribution, and the *red* and *green* curves represent the Wigner-Dyson distributions for GOE and GUE, respectively.

### Ratio of level spacing [5]

If $s_i$ is the difference in energy between the $ith$ and the $(i+1)th$ energy levels, i.e. $s_i = (E_{i+1} - E_i)$, then the ratio of level spacing $r_i$ is defined as,

$$r_i = \frac{min(s_i, s_{i+1})}{max(s_i, s_{i+1})}. \tag{18}$$

For random matrices with matrix elements chosen from Gaussian distributions, the eigenvalues will also have a Gaussian behaviour, and for a GOE, the probability distribution of the $r$ values comes out to be,

$$P(r) = \frac{1}{A_{GOE}} \frac{(r + r^2)}{(1 + r + r^2)^{5/2}}, \tag{19}$$

and,

$$P(r) = \frac{1}{A_{GUE}} \frac{(r + r^2)^2}{(1 + r + r^2)^4},$$

for GUE, where $A_{GOE}$ and $A_{GUE}$ are the normalising constants. These are shown in Fig. 2.

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