Spinning Conformal Correlator

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(Dated: September 29, 2023)

We review the embedding formalism for conformal field theories and we have shown an efficient way of computations with symmetric traceless operators of arbitrary spin. We encoded the tensors into polynomials using auxiliary vectors. This formalism is efficient for computing the tensor structures allowed in n-point conformal correlation functions of tensor operators. Conservation of tensor structure is also addressed using the differential equations in embedding space. Finally, the number of independent tensor structures of conformal correlators in d dimensions and the number of independent structures in scattering amplitudes of spinning particles in (d+1)-dimensional Minkowski space, matches.

1. INTRODUCTION

We know that Conformal Field Theories(CFT) put severe constraints on the correlation functions in the theory. These correlation functions are extremely important to study because they are the observables in the theory. Here we presented a formalism in general $d \geq 3$ Euclidean dimensions that will make CFT computations with tensor fields as easy as computations with scalars. Finally, we show that the number of tensor structures for three-point correlators of tensor operators is equal to the number of tensor structures for three particle S-matrix elements in one higher dimension.

2. EMBEDDING FORMALISM

In this paper, we consider CFT in $d \geq 3$ Euclidean dimensions, so that the conformal group is $SO(d+1,1)$. Now, we will develop the 'embedding formalism' which makes the nonzero spin case easier. The conformal group $SO(d+1,1)$ can be realized as the group of Lorentz symmetry in the *embedding space* $\mathbb{R}^{d+2}$[1]. On the $(d+1)$ dimensional light cone any vector will satisfy $P^2 = 0$. To connect this $(d+2)$ embedding space with our d dimensional physical space, we can quotient out the null cone by the rescaling $P \sim \lambda P$; $\lambda \in \mathbb{R}$. Hence, we can use this rescaling property to map from light cone coordinate, $P = (P^+, P^-, P^\mu)$ to the projected light cone coordinate, $P_x = \left(1, x^2, x^\mu\right)$.

Next we should establish the correspondence between fields on $\mathbb{R}^d$ and $\mathbb{R}^{d+2}$, which is done as follows[2]:

1. Defined on the cone $P^2 = 0$.
2. Homogeneous of degree $-\Delta$: $F_{A_1 \ldots A_l}(\lambda P) = \lambda^{-\Delta} F_{A_1 \ldots A_l}(P)$, $\lambda > 0$.
3. Symmetric and traceless.
4. Transverse: $(P \cdot F)_{A_2 \ldots A_l} \equiv P^A F_{A A_2 \ldots A_l} = 0$.

Projection of F onto the Poincaré section is given by,

$$f_{a_1 \ldots a_l}(x) = \frac{\partial P^{A_1}}{\partial x^{a_1}} \ldots \frac{\partial P^{A_l}}{\partial x^{a_l}} F_{A_1 \ldots A_l}(P_x)\,. \tag{2.1}$$

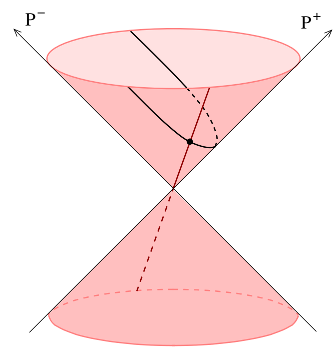

FIG. 1: Light cone in the embedding space; light rays are in one-to-one correspondence with physical space points. The Poincaré section (projected light cone) of the null cone is also shown where the metric is flat.

This operation has two important properties. First, any tensor proportional to P_A projects to zero. We will call such $SO(d+1,1)$ tensors *pure gauge*. Second, the projected tensor is traceless, as long as F is traceless and transverse.

3. ENCODING TENSORS BY POLYNOMIALS

Any symmetric, traceless tensor in $\mathbb{R}^d$ can be encoded by a d-dimensional polynomial using some auxiliary vector field:

$$f_{a_1 \ldots a_l}(x) \leftrightarrow f(x,z) \equiv f_{a_1 \ldots a_l} z^{a_1} \cdots z^{a_l}. \tag{3.1}$$

Due to tracelessness of the tensor we can restrict $z^2 = 0$. Hence, a given tensor is the same as a polynomial in z up to a vanishing z^2 term.
Now, define a differential operator[3]:

$$D_a = \left(h - 1 + z \cdot \frac{\partial}{\partial z}\right) \frac{\partial}{\partial z^a} - \frac{1}{2} z_a \frac{\partial^2}{\partial z \cdot \partial z} \qquad ; h = \frac{d}{2} \tag{3.2}$$

This differential operator will project the polynomial onto the tensor:

$$f_{a_1 \ldots a_l}(x) = \frac{(h-1)!}{l!(h-1+l)!} D_{a_1} \cdots D_{a_l} f(x,z) \tag{3.3}$$

Similarly, We can encode a general symmetric, traceless tensor in the embedding space by a $(d+2)$-dimensional polynomial

$$F_{A_1\ldots A_l}(P) \leftrightarrow F(P,Z) \equiv F_{A_1\ldots A_l}(P) Z^{A_1}\ldots Z^{A_l} \tag{3.4}$$

Using (2.1) we get on the section $Z_{z,x} \equiv (0, 2x\cdot z, z)$. Due to the tracelessness of the tensor, $Z^2 = 0$. Due to transversality of the tensor, $Z.P = 0$ and $P^2 = 0$ due to null cone restriction. So the polynomial for a given tensor is unique up to these vanishing conditions.
The following diagram relates embedding and physical tensors and polynomials[2]:

$$\begin{array}{ccc} F_{A_1\ldots A_l}(P) & \xrightarrow{(3.4)} & F(P,Z) \\ \downarrow{\scriptstyle (2.1)} & & \downarrow{\scriptstyle (2.1)} \\ f_{a_1\ldots a_l}(x) & \xleftarrow[(3.3)]{} & f(x,z) \end{array} \tag{3.5}$$

Here, $f(x,z) = F(P_x, Z_{z,x})$. For convenience, we choose to work with the $(d+2)$ dimensional polynomial structure. The physical space result always can always be retrieved using the steps in this diagram.

4. CORRELATION FUNCTIONS OF SPIN l PRIMARIES

In this section, we will compute the n-point correlation function of spin $l_1,\cdots,l_n$ primaries. For $n>3$, there are $n(n-3)/2$ independent conformally invariant cross-ratios u_a. For $n \le 3$ we won't encounter these cross ratios.
A generic n-point function in the embedding space, $G_{\chi_1,\ldots,\chi_n}(P_1,\cdots,P_n)$ is isomorphic to a polynomial, $\tilde{G}_{\chi_1,\ldots,\chi_n}(P_1,Z_1\cdots,P_n,Z_n)$. The lorentz invariance, transversality and homogeneity of the polynomial in the embedding space ensure that,

$$\tilde{G}_{\chi_1,\ldots,\chi_n}(\{P_i;Z_i\}) = \tilde{G}_{\chi_1,\ldots,\chi_n}(\{P_i.P_j, Z_i.Z_j, P_i.Z_j\})$$

Now, we claim that $\tilde{G}_{\chi_1,\ldots,\chi_n}$ can be expanded as:

$$\tilde{G}_{\chi_1,\ldots,\chi_n} = \prod_{i<j}^{n} P_{ij}^{-\alpha_{ij}} \sum_k f_k(u_a)\, Q^{(k)}_{\chi_1,\ldots,\chi_n}(\{P_i;Z_i\}) \tag{4.1}$$

$$\text{Where,}\quad \alpha_{ij} = \frac{\tau_i+\tau_j}{n-2} - \frac{1}{(n-1)(n-2)}\sum_{k=1}^{n}\tau_k \quad ;n>2$$
$$P_{ij} = P_i.P_j \qquad ,\tau_i = \Delta_i + l_i$$

(Here, we will show an explicit calculation for $n=3$ in this section. Example for $n=2$-point function is given in the (appendix A).)
To justify our claim in (4.1), we choose the pre-factor, $Q^{(k)}$ have weight l_i in each point P_i. They are also identically transverse:

$$\begin{aligned} &Q^{(k)}_{\chi_1,\ldots,\chi_n}(\{\lambda_i P_i; \alpha_i Z_i + \beta_i P_i\}) \\ &\quad = Q^{(k)}_{\chi_1,\ldots,\chi_n}(\{P_i;Z_i\}) \prod_i (\lambda_i\alpha_i)^{l_i} \end{aligned} \tag{4.2}$$

Now, we claim that $Q^{(k)}_{\chi_1,\ldots,\chi_n}(\{P_i;Z_i\})$ can be built out of two types of basic building blocks, $H_{ij}, V_{i,jk}$, satisfying the condition in (4.2).

$$H_{ij} = -2\big[(Z_i\cdot Z_j)(P_i\cdot P_j) - (Z_i\cdot P_j)(Z_j\cdot P_i)\big] \tag{4.3}$$
$$V_{i,jk} = \frac{(Z_i\cdot P_j)(P_i\cdot P_k) - (Z_i\cdot P_k)(P_i\cdot P_j)}{(P_j\cdot P_k)}\,, \tag{4.4}$$

Note that, $H_{ij} = H_{ji}$ & $V_{i,jk} = -V_{i,kj}$.
For each i all the $V_{i,jk}$ are not linearly independent. For simplicity, we will show an example of a 3-point function, where we have only one $V_{i,jk}$ for each i.

4.1. Example:

A 3-point function $\tilde{G}_{\chi_1,\chi_2,\chi_3}(\{P_i,Z_i\}$ of spin l_1,l_2,l_3 primaries can be written following the above rules as,

$$\tilde{G}_{\chi_1,\chi_2,\chi_3}(P_i;Z_i) = \frac{Q_{\chi_1,\chi_2,\chi_3}(\{P_i;Z_i\})}{(P_{12})^{\frac{\tau_1+\tau_2-\tau_3}{2}}(P_{23})^{\frac{\tau_2+\tau_3-\tau_1}{2}}(P_{31})^{\frac{\tau_3+\tau_1-\tau_2}{2}}} \tag{4.5}$$

According to above discussion the general solution for Q_{χ_1,χ_2,χ_3} can be written as a linear combination of

$$\prod_i V_i^{m_i} \prod_{i<j} H_{ij}^{n_{ij}}\,. \tag{4.6}$$

Since Q must have degree l_i in each Z_i, the exponents must satisfy the three constraints

$$m_i + \sum_{j\neq i} n_{ij} = l_i\,. \tag{4.7}$$

Now, this is a counting problem to find all the inequivalent structures in the 3-point function.
For 3-point function of identical spin-2 operators we have 5 inequivalent structures,

$$\begin{aligned} &A_1 = (V_1.V_2.V_3)^2, \qquad A_2 = H_{12}V_1.V_2.V_3^2 + \cdots \\ &A_2 = H_{12}.H_{13}.V_2.V_3 + \cdots\,, \quad A_3 = H_{12}H_{13}H_{23} \\ &A_4 = H_{12}^2 V_3^2 + \cdots \end{aligned}$$

Hence, 3 point function of identical spin-2 operators in embedding space is,

$$\tilde{G}_{\chi_1,\chi_2,\chi_3}(\{P_i;Z_i\}) = \frac{\alpha.A_1 + \beta.A_2 + \gamma.A_3 + \rho.A_4 + \sigma.A_5}{\prod\limits_{\substack{i<j,k\neq i,j\\ i,j,k=1}}^{3} (P_{ij})^{(\tau_i+\tau_j-\tau_k)/2}} \tag{4.8}$$

(An interesting way of counting these conformally invariant structures has been shown by an example in (appendix C).)
Note that, for n-point correlation function, the $2n$ vectors Z_i and P_i can not be linearly independent in the $(d+2)$-dimensional embedding space if $n > \frac{d}{2}+1$. Here we have considered $n < \frac{d}{2}+1$ to write down the independent conformally invariant structures for a given correlator from (4.7). For $n > \frac{d}{2}+1$, there should be a reduction in the number of possible conformally invariant structures[2].

5. CONSERVED TENSORS

In unitary CFTs, the dimensions of spin l primaries must satisfy the unitarity bound : $\Delta \geq l+d-2 \quad (l \geq 1)$. When Δ takes the lowest value allowed by this bound for a given l, the corresponding primary field is conserved. Physically important examples of such fields are the stress tensor ($l = 2$) and global symmetry currents ($l = 1$).
Let us begin by considering the conservation condition for a spin l dimension Δ primary:

$$(\partial \cdot f)^{a_2 \ldots a_l} \equiv \frac{\partial}{\partial x^{a_1}} f^{a_1 a_2 \ldots a_l}(x) = 0 \tag{5.1}$$

We have shown the following result in (appendix B)

$$(\partial \cdot f)_{a_2 \ldots a_l}(x) = \frac{\partial P^{A_2}}{\partial x^{a_2}} \ldots \frac{\partial P^{A_l}}{\partial x^{a_l}} R_{A_2 \ldots A_l}(P_x)\,, \tag{5.2}$$

Where,

$$R_{A_2 \ldots A_l}(P) = \left[\frac{\partial}{\partial P_{A_1}} - \frac{1}{P \cdot \bar{P}}(\bar{P} \cdot \frac{\partial}{\partial P})P^{A_1} - (l+d-2-\Delta)\frac{\bar{P}^{A_1}}{P \cdot \bar{P}}\right] \tag{5.3}$$

Where, $\bar{P} = (0, 2, 0)$.
Now we see what is special about $\Delta = l+d-2$ from (5.3): precisely for this dimension R becomes an $SO(d+1, 1)$ invariant tensor. Hence, for $\Delta = l+d-2$, and only for this dimension, the conservation condition $\partial \cdot f = 0$ can be imposed in a way that is consistent with the conformal symmetry.
All the tensors are encoded to polynomial structures in the embedding space. So, we have to encode the tensor R also via the identically transverse function $\tilde{R}(P, Z)$.
It can be shown that[2],

$$\tilde{R}(P, Z) = \frac{1}{l(h+l-2)}(\partial \cdot D)F(P, Z) \tag{5.4}$$

where $\partial \cdot D \equiv \frac{\partial}{\partial P_M} D_M$. And D_M is the differential operator in Z defined in Eq. (3.2).
Let us consider the simplest nontrivial example of a three-point function between two vector currents at points x_1 and x_2 and a scalar operator at x_3, $\langle v^1_a(x_1) v^2_b(x_2) \phi(x_3)\rangle$. Here we assume that ϕ has dimension Δ, while v's necessarily have dimension $d-1$.

According to the results of section 4.1, the embedding function encoding this three-point function (for the symmetric case under current exchange) has the form

$$\tilde{G}(P_1, P_2, P_3; Z_1, Z_2) = \frac{\alpha V_1 V_2 + \beta H_{12}}{(P_{12})^{d-\frac{\Delta}{2}} (P_{13})^{\frac{\Delta}{2}} (P_{23})^{\frac{\Delta}{2}}}\,, \tag{5.5}$$

For conservation of currents we can compute the divergence at P_1 and drop the terms of $O(Z_1^2, Z_1 \cdot P_1)$, we find the result

$$(\partial_{P_1} \cdot D_{Z_1})\,\tilde{G} \to \left(\frac{d}{2} - 1\right) \frac{(\alpha(d-1-\Delta) + \beta\Delta) \cdot V_2}{(P_{12})^{d-\frac{\Delta}{2}} (P_{13})^{\frac{\Delta}{2}} (P_{23})^{\frac{\Delta}{2}}}\,. \tag{5.6}$$

Hence, conservation of currents implies that,

$$\alpha(d-1-\Delta) + \beta\Delta = 0. \tag{5.7}$$

This conservation condition constraint the 3-point correlator,$\langle v^1_a(x_1) v^2_b(x_2) \phi(x_3)\rangle$ to have one independent tensor structure instead of two.

6. S-MATRIX RULE FOR COUNTING STRUCTURES

We propose the following generalization: The number of independent structures in a three-point function containing operators of spins $\{l_1, l_2, l_3\}$ is equal to the number of independent on-shell scattering amplitudes for particles of spins $\{l_1, l_2, l_3\}$ in $d+1$ flat Minkowski dimensions. The particles should be taken massless or massive depending on whether or not the corresponding operator is conserved.
To demonstrate this, let us first consider the case of a scattering amplitude between 3 massive particles of arbitrary spin. It is a Lorentz invariant function of the momentum p_i and polarization tensor ζ_i of each particle. Here we can treat the polarization tensor, ζ_i as the auxiliary vector z_i, mentioned in section 3. Moreover, the transversality condition $(p_i)_{\mu_1}\zeta_i^{\mu_1 \ldots \mu_{l_1}} = 0$ translates to $z_i \cdot p_i = 0$.
Therefore, we must count polynomials such that

$$S(\{P_i; \lambda_i Z_i\}) = \prod_{i=1}^{3} \lambda_i^{l_i} S(\{P_i; Z_i\}) \tag{6.1}$$

where $z_i \cdot p_i = 0 \quad p_1 + p_2 + p_3 = 0$ and $p_i^2 = -M_i^2$
Therefore, the general solution is a linear combination of

$$S(n_{12}, n_{13}, n_{23}) = \prod_{i<j}^{n} ((z_i \cdot p_j)^{m_i} \cdot (z_i \cdot z_j)^{n_{ij}}) \tag{6.2}$$

$$\text{Where,} \qquad m_i = l_i - \sum_{j \neq i} n_{ij} \geq 0\,. \tag{6.3}$$

Since this is the same condition as Eq. (4.7), the number of solutions is given by exactly the same combinatorial problem that we solved for non-conserved CFT three-point functions.

Let us now study massless particles. In this case, the scattering amplitude must be invariant under the infinitesimal gauge transformation

$$\zeta_{\mu_1 \ldots \mu_l} \to \zeta_{\mu_1 \ldots \mu_l} + p_{(\mu_1}\Lambda_{\mu_2 \ldots \mu_l)}\,. \tag{6.4}$$

This corresponds to invariance under

$$z_\mu \to z_\mu + \epsilon\, p_\mu \tag{6.5}$$

to first order in ϵ. The problem of finding gauge invariant 3-particle scattering amplitudes is then reduced to finding linear combinations of the structures (6.2) that are invariant under (6.5) to first order in ϵ.
An explicit calculation in (appendix E) shows that the number of possible scattering amplitudes between 3 massless higher spin particles is $1 + \min(l_1, l_2, l_3)$. In general this will match the counting of conformal three-point functions of conserved tensors in $d \geq 4$[4].

Appendix A: 2 point correlation of spin 1 operators:

From the discussion of section 4, we can understand that the two-point correlation of two operators can be built using H_{12}^n only. Hence, two operators should have the same spin (l) as n ,for nonzero correlation.
In the light cone two point function of two operators can be encoded using polynomial,

$$\begin{aligned} &G_\chi(P_1,P_2;Z_1,Z_2) \\ &= Z_1^{A_1}\cdots Z_1^{A_l} Z_2^{B_1}\cdots Z_2^{B_2}\, G_{A_1\ldots A_l,B_1\ldots B_l}(P_1,P_2) \end{aligned} \quad (A.1)$$

According to section 4 discussion this polynomial has to satisfy certain condition,

$$G_\chi(\{\lambda_i P_i;\beta_i Z_i+\alpha_i Z_i\}) = (\lambda_i)^{-\Delta}(\beta_i)^l G_\chi(\{P_i;Z_i\}) \quad (A.2)$$

This condition can be satisfied by claiming,

$$G_\chi(P_1,P_2;Z_1,Z_2) = const\,\frac{(H_{12})^l}{P_{12}^{\Delta+l}} \quad (A.3)$$

For $l=1$ we will have,

$$\begin{aligned} &G_\chi(P_1,P_2;Z_1,Z_2) \\ &= const\,\frac{-2\big[(Z_1\cdot Z_2)(P_1\cdot P_2)-(Z_1\cdot P_2)(Z_2\cdot P_1)\big]}{(P_1\cdot P_2)^{\Delta+1}} \end{aligned} \quad (A.4)$$

Here,

$$\begin{aligned} P_1^\mu &= (1,x_1^2,x_1^\mu), \quad P_2^\mu = (1,x_2^2,x_2^\mu) \\ Z_1^\mu &= \frac{\partial P_1^\nu}{\partial x_1^\mu}(z_1)_\nu = (0,2x_1\cdot z_1,z_1^\mu) \\ Z_2^\mu &= \frac{\partial P_2^\nu}{\partial x_2^\mu}(z_2)_\nu = (0,2x_2\cdot z_2,z_2^\mu) \end{aligned}$$

Substituting these in the (A.4) we get,

$$G_\chi(x_1,x_2;z_1,z_2) = const\,\frac{\big[(\frac{-1}{2})(z_1\cdot z_2)(x_2-x_1)^2-(x_1\cdot z_2-x_2\cdot z_2)(x_2\cdot z_1-x_1\cdot z_1)\big]}{(x_2-x_1)^{2\Delta+2}} \quad (A.5)$$

Here we have projected our lightcone result for two point function,(A.4) onto the Poincaré section. Now we can extract the tensorial structure from this polynomial using the differential operator defined in (3.3). Hence,

$$\begin{aligned} G_\chi(x_1,x_2) &= \langle V_1^\mu(x_1)\,V_2^\nu(x_2)\rangle \\ &= \frac{1}{(h-1)^2} D_{z_1}^\mu D_{z_2}^\nu\, G_\chi(x_1,x_2;z_1,z_2) \end{aligned} \quad (A.6)$$

Here D_{z_1},D_{z_2} are operators in (3.2) with z_1 and z_2.
Now, equating (A.6) we get,

$$G_\chi(x_1,x_2) = const\,\frac{\big[(\frac{1}{2})(x_2-x_1)^2\,\delta^{\mu\nu}+(x_1^\mu-x_2^\mu)(x_2^\nu-x_1^\nu)\big]}{(x_2-x_1)^{2\Delta+2}} \quad (A.7)$$

This result matches the two point correlation of vector operators mentioned in [5].

Appendix B: Derivation of $R_{A_2\ldots A_l}(P)$

In the projected light cone we have the following result,

$$\frac{\partial P^A}{\partial x^c} = (0,2x_c,\delta_c^a) \quad (B.1)$$

$$\text{Hence,}\quad \frac{\partial}{\partial x^b}\Big(\frac{\partial P^A}{\partial x^c}\Big) = \bar{P}^A\,\delta_{bc} \quad (B.2)$$

Here, $\bar{P}^A = (0,2,0)$.
Now, we claim the following result

$$K^{AB} \equiv \delta^{ab}\frac{\partial P^A}{\partial x^a}\frac{\partial P^B}{\partial x^b} = \eta^{AB}+P^A\bar{P}^B+P^B\bar{P}^A \quad (B.3)$$

This claim can be justified by contracting both sides with P_A, P_B.
Now, using (5.1) condition on (2.1) we get,

$$\begin{aligned} &\frac{\partial}{\partial x_{a_1}} f_{a_1a_2\ldots a_l}(x) = 0 \\ &\implies \frac{\partial}{\partial x_{a_1}}\Big(\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}F_{A_1\ldots A_l}(P)\Big) = 0 \\ &\implies \frac{\partial}{\partial x_{a_1}}\Big(\frac{\partial P^{A_1}}{\partial x^{a_1}}\Big)\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}F_{A_1\ldots A_l}(P) \\ &\quad + \sum_{i\neq 1}\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial}{\partial x_{a_1}}\Big(\frac{\partial P^{A_i}}{\partial x^{a_i}}\Big)\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}F_{A_1\ldots A_l}(P) \\ &\quad + \frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}\frac{\partial P^B}{\partial x_{a_1}}\frac{\partial F_{A_1\ldots A_l}(P)}{\partial P^B} = 0 \\ &\implies d\cdot\frac{\partial P^{A_2}}{\partial x^{a_2}}\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}\bar{P}^{A_1}F_{A_1\ldots A_l}(P) \\ &\quad + \sum_{i\neq 1}\frac{\partial P^{A_1}}{\partial x^{a_1}}\cdots\bar{P}^{A_i}\delta_{a_i}^{a_1}\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}F_{A_1\ldots A_l}(P) \\ &\quad + \frac{\partial P^{A_2}}{\partial x^{a_2}}\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}K^{A_1B}\frac{\partial F_{A_1\ldots A_l}(P)}{\partial P^B} = 0 \end{aligned}$$

$$\Longrightarrow -\frac{d}{P\cdot\bar{P}}\cdot\frac{\partial P^{A_2}}{\partial x^{a_2}}\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}\bar{P}^{A_1}F_{A_1\ldots A_l}(P)$$
$$-\frac{1}{P\cdot\bar{P}}\frac{\partial P^{A_2}}{\partial x^{a_2}}\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}\bar{P}^{A_1}\Big[F_{A_2A_1\ldots A_l}(P)$$
$$+F_{A_3A_2A_1\ldots A_l}(P)+\cdots+F_{A_lA_2\ldots A_1}(P)\Big]$$
$$+\frac{\partial P^{A_2}}{\partial x^{a_2}}\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}K^{A_1B}\frac{\partial F_{A_1\ldots A_l}(P)}{\partial P^B}=0$$
$$\Longrightarrow \frac{\partial P^{A_2}}{\partial x^{a_2}}\cdots\frac{\partial P^{A_l}}{\partial x^{a_l}}R_{A_2\ldots A_l}(P)=0$$

Here,

$$R_{A_2\ldots A_l}(P)=\Bigg[\frac{\partial}{\partial P_{A_1}}-\frac{1}{P\cdot\bar{P}}(\bar{P}\cdot\frac{\partial}{\partial P})P^{A_1} -(l+d-2-\Delta)\frac{\bar{P}^{A_1}}{P\cdot\bar{P}}\Bigg] \tag{B.4}$$

Note that, $P\cdot\bar{P}=-1$ and we have divided this quantity in the derivation to maintain the homogeneity in P. In the second last line of the derivation we have used the expression of K^{A_1B} and we have also used the symmetry property of $F_{A_lA_2\ldots A_1}(P)$.
The derivation of $\tilde{R}(P,Z)$ is shown in section 5.2 of [2].

Appendix C: Visualization of Conformally invariant structure in embedding space

We have seen that the n-point correlation function's numerator is fixed by the fundamental building blocks $H_{ij}, V_{i,jk}$ which satisfy (4.2). All the conformally invariant allowed structures in the embedding space can be calculated for a n-point correlation function using (4.7). Note that we have derived (4.7) for 3-point function only, but this treatment can be followed generally, and there we will get a similar expression.
Now, Instead of solving (4.7) we can visualize a schematic representation of one of the tensor structures appearing in the (spin 5)-(spin 3)-(spin 7) three-point function as Fig.2.

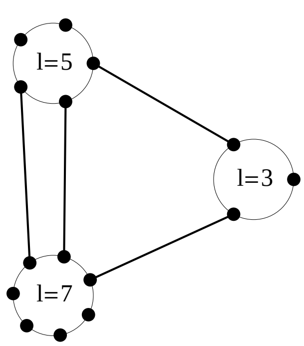

FIG. 2

In this diagram, V_i's are represented as disconnected dots at the vertices and H_{ij}'s as lines joining the vertices.

Appendix D: Conservation of 3-point function for (spin 2)-(spin 2)-(spin 0)

According to the discussion of of section 4.1, the 3-point function of spin-2, spin-2, spin-0 operators has 3 inequivalent polynomial structures,

$$\tilde{G}(\{P_i;Z_i\})=\frac{\alpha.V_1^2V_2^2+\beta.H_{12}^2+\gamma.H_{12}V_1V_2}{(P_{12})^{d+2-\frac{\Delta}{2}}(P_{13})^{\frac{\Delta}{2}}(P_{23})^{\frac{\Delta}{2}}} \tag{D.1}$$

Here, we have assumed the spin two operators to be the same with scaling dimension d and the scalar operator has scaling dimension Δ.
This structure is symmetric under the exchange of $P_1\leftrightarrow P_2$ and $Z_1\leftrightarrow Z_2$ because under the exchange H_{ij} is symmetric and V_i is antisymmetric.
For conservation of spin-2 operators, we can compute the divergence at P_1 and drop the terms of $O(Z_1^2, Z_1\cdot P_1)$, we find the result using (5.4)

$$(\partial_{P_1}\cdot D_{Z_1})\,\tilde{G} =\frac{(-2\alpha+2d\alpha-2\Delta\alpha-2\gamma)V_1V_2^2-(2\Delta\beta+\gamma)H_{12}V_2}{(P_{12})^{d+2-\frac{\Delta}{2}}\,(P_{13})^{\frac{\Delta}{2}}\,(P_{23})^{\frac{\Delta}{2}}}\,. \tag{D.2}$$

Hence the conservation of spin-2 stress tensor implies that,

$$2\alpha+2d\alpha+2\Delta\alpha+2\gamma=0 \tag{D.3}$$
$$2\Delta\beta+\gamma=0 \tag{D.4}$$

From (D.3) & (D.4), we can say that the three independent tensor has been turned into only one independent tensor structure for 3 point function of two conserved spin-2 operator with one scalar operator. This matches the result in [4].

Appendix E: Scattering amplitude for massless particles

For massless particles scattering, we have additional gauge symmetry (6.5) to invoke in (6.2).
Using (6.5) in (6.2) we get,

$$\delta_1 S(n_{12},n_{13},n_{23})=\epsilon_1\Big[n_{13}\,S_1(n_{12},n_{13}-1,n_{23}) -n_{12}\,S_1(n_{12}-1,n_{13},n_{23})\Big] \tag{E.1}$$

$$\delta_2 S(n_{12},n_{13},n_{23})=\epsilon_2\Big[n_{12}\,S_2(n_{12}-1,n_{13},n_{23}) -n_{23}\,S_2(n_{12},n_{13},n_{23}-1)\Big] \tag{E.2}$$

$$\delta_3 S(n_{12}, n_{13}, n_{23}) = \epsilon_3 \Big[n_{23}\, S_3(n_{12}, n_{13}, n_{23}-1) - n_{13}\, S_3(n_{12}, n_{13}-1, n_{23}) \Big] \tag{E.3}$$

where S_i is given by the same expression as S but with $l_i \to l_i - 1$.
Let's take an ansatz of the following linear combination, which will be invariant under the gauge transformation mentioned above.

$$\sum_{i=0}^{k} a_i\, S(i, k-i, n_{23}) \tag{E.4}$$

to impose gauge invariance for particle 1. We then find that

$$\sum_{i=0}^{k} \left(a_i\, i\, S_1(i-1, k-i, n_{23}) - a_i\, (k-i)\, S_1(i, k-i-1, n_{23}) \right) = \sum_{i=1}^{k} \left(a_i\, i - a_{i-1}\, (k-i+1) \right)\, S_1(i-1, k-i, n_{23}) = 0, \tag{E.5}$$

which fixes all the coefficients up to an overall normalization,

$$a_i = \frac{k-i+1}{i} a_{i-1} = \frac{k!}{i!(k-i)!} a_0\ . \tag{E.6}$$

Notice that this solution only exists for $k \le l_1$.

Imposing gauge invariance also on particle 2, we find the amplitude

$$T_k = \sum_{i=0}^{k} \sum_{j=0}^{k-i} \frac{k!}{i!j!(k-i-j)!}\, S(i, j, k-i-j)\ . \tag{E.7}$$

Gauge invariance of particle 3 is automatic. Note that this solution only exists for k smaller (or equal) than all the spins l_i. Therefore, the number of possible scattering amplitudes between 3 massless higher spin particles is

$$1 + \min(l_1, l_2, l_3)\ .$$

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