An Introduction to Liouville Field Theory

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In this article, I introduced the Liouville theory as a conformal field theory point of view. I found equation of motion, the primary operator, calculated the stress-energy tensor and how they change in cylindrical coordinates. How the picture changes in quantum theory has also been discussed. I talked about three point functions and DOZZ formula at the end. Some future directions are given in later section.

INTRODUCTION

Over the last few decades, Liouville field theory drew huge attention of researchers in the advancement of critical and non-critical string theory and in general relativity. It can be shown that for quantizing 2-d gravity, the main problem eventually boils down to, finding characteristics of Liouville field theory. As string theory is all about working on 2-d worldsheets, there is a direct coupling between Liouville fields and the worldsheet geometry. Also, in phase transition of some statistical models, it has profound applications, which make the theory worth studying. Liouville field theory (LFT) also is the simplest non-minimal conformal field theory (CFT) with a continuous spectrum of primary fields that serves as a prototype to develop techniques that can be helpful in the study of more complicated CFTs. In this short article, I will provide an introduction to Liouville theory from the angle of a conformal field theory. I will start with 2-d gravity to provide a motivation to study LFTs.

2-DIMENSIONAL GRAVITY

The action for a 2-d gravity can be written as

$$A[g_{ab}] = \mu \int \mathrm{d}^2\sigma \sqrt{g} + k \int \mathrm{d}^2\sigma \sqrt{g} R$$
$$+ \, l \int \mathrm{d}^2\sigma \sqrt{g}[R^2 + \nabla^a \nabla_a R + \text{other higher degree terms}].$$

The first term gives us the total surface area of the manifold. The second term using Gauss-Bonet theorem, evaluates to

$$\int \mathrm{d}^2\sigma \sqrt{g} R = 4\pi\chi,$$

where $\chi = 2(1-g)$, g =genus of the manifold. So, first two terms are surface terms, are related to topology of the manifold. The last terms are very small to show any significant influence and also very complicated to work with. In 2-d we don't get non-trivial dynamical vacuum solutions by generalizing Einstein's gravity. But, we can always use a coordinate transformation locally, to make the metric in the following form,

$$g_{ab}(\sigma) = e^{\phi(\sigma)}\delta_{ab}. \tag{1}$$

Here, for simplicity we are working on a manifold with Euclidean signature. The Ricci scalar can be calculated from this metric,

$$R(\sigma) = -e^{\phi(\sigma)}(\partial_0^2 + \partial_1^2)\phi(\sigma),$$
$$\implies (\partial_0^2 + \partial_1^2)\phi(\sigma) + R(\sigma)e^{\phi(\sigma)} = 0.$$

The last equation matches with familiar Liouville equation,

$$(\partial_0^2 + \partial_1^2)\phi(\sigma) + \Lambda e^{\phi(\sigma)} = 0. \tag{2}$$

So in this gauge (1), Liouville field $\phi(\sigma)$ specifies a manifold with constant curvature $R(\sigma) = \Lambda$ (a constant). To examine properties of Liouville field $\phi(\sigma)$, we need to find an action which is diffeomorphism invariant and in conformal gauge (1) produces Liouville equation. It's difficult to find such action. We can at the best do, is to factorise the physical metric as

$$g_{\mu\nu} = e^{2b\phi}h_{\mu\nu}$$

and write an action with $h_{\mu\nu}$ metric,

$$S = \frac{1}{4\pi}\int \mathrm{d}^2\sigma \sqrt{h}\big(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + QR\phi + 4\pi\mu e^{2b\phi}\big)$$

and at the end take $h_{\mu\nu} = \delta_{\mu\nu}$. The validity of this action comes from the fact that it indeed produces Liouville equation in conformal gauge. We will see that later. Here, R is the Ricci scalar associated with the background metric $h_{\mu\nu}$ and $Q = \frac{1}{b}$, is a constant, called the background charge. b and $\mu(>0)$ are arbitrary constants.

LIOUVILLE THEORY

The classical Liouville field theory is described by this action,

$$S = \frac{1}{4\pi}\int \mathrm{d}^2\sigma \sqrt{h}\big(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + QR\phi + 4\pi\mu e^{2b\phi}\big) \tag{3}$$

Here, we considered the theory is coupled to 2-d gravity with non-minimal coupling. The physical metric on 2-d surface is $g_{\mu\nu}$ and as the action must be diffeomorphism invariant, it will be a gauge theory. We can fix the gauge by choosing the conformal gauge condition,

$$g_{\mu\nu} = e^{2b\phi} h_{\mu\nu} \tag{4}$$

where $h_{\mu\nu}$ is fixed, is called fiducial or non-physical metric. In conformal gauge, the action have another symmetry, called Weyl symmetry

$$h_{\mu\nu} \to e^{2\omega} h_{\mu\nu}, \qquad \phi \to \phi - Q\omega. \tag{5}$$

I note here that this Weyl symmetry is not fundamental and is emerging from the theory in conformal gauge. Diffeomorphism and Weyl symmetry makes the theory conformally invariant[4].

The central charge of the theory can be calculated using Coulomb gas representation

$$c_L = 6Q^2.$$

Because of invariane under (5), we can use Weyl transformation to make the metric locally flat,

$$h_{\mu\nu} = \delta_{\mu\nu}.$$

Then the action (3), becomes

$$S = \frac{1}{4\pi} \int \mathrm{d}^2\sigma \big(\partial_\mu\phi\partial^\mu\phi + QR\phi + 4\pi\mu e^{2b\phi}\big).$$

We keep the Ricci curvature even if $R = 0$ because it will have contributions to energy-momentum tensor.

EQUATION OF MOTION

We can now calculate equation of motion for the field ϕ and the stress-energy tensor. For that, we vary the action (3) w.r.t. $h_{\mu\nu}$ and ϕ,

$$\begin{aligned}\delta_h S = \frac{1}{4\pi} \int \mathrm{d}^2\sigma\sqrt{h}\delta h^{\mu\nu}\Big[&-\frac{1}{2}h_{\mu\nu}\big(h^{\rho\sigma}\partial_\rho\partial_\sigma\phi + QR\phi \\ &+ 4\pi\mu e^{2b\phi}\big) + \big(\partial_\mu\phi\partial_\nu\phi + QR_{\mu\nu}\phi \\ &+ Q(h_{\mu\nu}\Delta\phi - \nabla_\mu\nabla_\nu\phi)\big)\Big] \\ &(\text{as we know, } \delta R = R_{\mu\nu}\delta h^{\mu\nu} \\ &- \nabla_\mu\nabla_\nu\delta h^{\mu\nu} + h_{\mu\nu}\nabla^2\delta h^{\mu\nu}), \\ \delta_\phi S = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\sqrt{h}\delta\phi(&-2\Delta\phi + QR + 8\pi\mu b e^{2b\phi}).\end{aligned}$$

The equation of motion for ϕ gives

$$QR[h] - 2\Delta\phi = -8\pi\mu b e^{2b\phi}. \tag{6}$$

Here, $\Delta = \frac{1}{\sqrt{h}}\partial_\mu(\sqrt{h}h^{\mu\nu}\partial_\nu\phi)$ is Laplace-Beltrami operator. For $h_{\mu\nu} = \delta_{\mu\nu}$ the last equation reduces to

$$\partial_\mu\partial^\mu\phi = 4\pi\mu b e^{2b\phi},$$

which has exactly the same form of Liouville equation (2). We now compute the stress-energy tensor using

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{h}}\frac{\delta S}{\delta h^{\mu\nu}}.$$

Which gives

$$\begin{aligned}T_{\mu\nu} = -\big(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}h_{\mu\nu}h^{\rho\sigma}\partial_\rho\partial_\sigma\phi\big) \\ + Q\big(\nabla_\mu\nabla_\nu\phi - h_{\mu\nu}\Delta\phi\big) + 2\pi\mu h_{\mu\nu}e^{2b\phi}.\end{aligned}$$

Trace of this tensor is

$$T = h^{\mu\nu}T_{\mu\nu} = -Q\Delta\phi + 4\pi\mu e^{2b\phi}.$$

Using (6), we have

$$T = -\frac{Q^2}{2}R. \tag{7}$$

So, on-shell in conformal gauge the trace is zero. To use the techniques of usual CFT, we use complex coordinates $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 - i\sigma^2$ We recall that the metric will take the form

$$\mathrm{d}s^2 = h_{\mu\nu}\mathrm{d}\sigma^\mu\mathrm{d}\sigma^\nu = h_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}.$$

Under the change, $z \to w(z)$ and $\bar{z} \to \bar{w}(\bar{z})$ metric becomes $h_{w\bar{w}} = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}h_{z\bar{z}}$. We get the conformal factor $\omega = \ln\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|$. As a consequence the Liouville field transforms as

$$\phi' = \phi - Q\ln\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right| \tag{8}$$

This is not a primary field. We can see that the primary fields will be $e^{2a\phi}$ since

$$e^{2a\phi(z)} \to e^{2a\phi'(w)} = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^{2aQ} e^{2a\phi(z)}$$

We consider only the holomorphic part. For each value of a we have a primary field. The spectrum is continuous so Liouville theory is not a minimal model.

Let's compactify one dimension to put the CFT on the cylinder

$$\sigma^i = (\tau, \sigma)$$
$$\tau \in \mathbb{R}, \quad \sigma \in [0, 2\pi].$$

The cylindrical coordinates will be $w = \tau + i\sigma$. We again map it to a plane $z = e^w = e^{\tau + i\sigma}$. The metric in z coordinate becomes

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z}.$$

In this coordinate the stress energy tensor takes the form

$$T = -(\partial\phi)^2 + Q\partial^2\phi + 2\pi\mu e^{2b\phi}.$$

On-shell, it will be,

$$T = -\frac{1}{2}(\partial\phi)^2 + Q\partial^2\phi.$$

If we had put $R = 0$ initially then we would have missed the last term. Using (8), the primary fields $v_b = e^{2b\phi}$ on the cylinder transform to $v_b = e^{(2b\phi-a)}$.

All the above discussions were done for classical Liouville theory. If we try to quantize the theory, many symmetries break down. The traceless energy-momentum tensor in (7) gains trace in quantum theory because of trace anomaly. The quantum expectation value becomes

$$\langle T^\mu_\mu \rangle = -\frac{c_L}{12}R = -\frac{1}{12} - \frac{Q^2}{2}R.$$

The value of Q also changes,

$$Q = b + \frac{1}{b}.$$

Because of this, the theory starts to have an additional symmetry $b \to \frac{1}{b}$. The signatures of quantum theory can be found in this paper [2].

The most important aspect about a CFT, that is finding the correlation functions, becomes very difficult for Liouville theory. To obtain a closed form of three point function took many years to succeed.

THREE POINT FUNCTION

The form of three point function is completely determined by the conformal symmetry,

$$G_{\alpha_1\alpha_2\alpha_3}(z_1, z_2, z_3) = |z_{12}|^{2\gamma_3}|z_{31}|^{2\gamma_2}|z_{32}|^{2\gamma_1}c(\alpha_3, \alpha_2, \alpha_1)$$

where

$$\gamma_1 = \Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{\alpha_3}; \qquad \gamma_2 = \Delta_{\alpha_2} - \Delta_{\alpha_3} - \Delta_{\alpha_1}; \\ \gamma_3 = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}.$$

Determination of $c(\alpha_3, \alpha_2, \alpha_1)$ depends on particular theories. In two papers[1, 8] by *D-O-Z-Z*, for the first time, an explicit formula for this functions were proposed. These were given by

$$c(\alpha_3,\alpha_2,\alpha_1) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-\alpha_1-\alpha_2-\alpha_3}{b}} \frac{\gamma_0\gamma_b(2\alpha_1)\gamma_b(2\alpha_2)\gamma_b(2\alpha_3)}{\gamma_b(\alpha_1+\alpha_2+\alpha_3-Q)\gamma_b(\alpha_1+\alpha_2-\alpha_3)} \times \frac{1}{\gamma_b(\alpha_1+\alpha_3-\alpha_2)\gamma_b(\alpha_2+\alpha_3-\alpha_1)}.$$

Here, $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and γ_b functions are defined as

$$\log\,\gamma_b(x) = \int_0^\infty \frac{\mathrm{d}t}{t}\left[\left(\frac{Q}{2} - x\right)^2 e^{-t} - \frac{\sinh^2\left(\frac{Q}{2} - x\right)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right].$$

There are many important properties of these γ_b functions that are worth noticing. These properties can be found in [6], also in [5].

We can move on to calculate four-point functions. Though a compact form of the function have not been found yet, great amount of work has been done on it. Interested readers can refer [5] for more information.

FUTURE WORKS

This was a very short introduction to Liouville theory, discussing only basic topics. Many important aspects of Liouvillle theory could have been covered in great detail such as quantizing the theory, proving DOZZ formula, using conformal blocks to find four point functions, conformal bootstrap procedure etc. Applications to some statistical models such as 2-d Ising model have not been done here. They are reserved for future works.

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