# Conformal Quantum mechanics

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In this report, we will look into the d=1 CFT (0+1) or, Conformal quantum mechanics. We observe that the generators of the conformal group, which is SO(2,1), don't exactly annihilate the ground state and are non-primary nature. Instead of using the states of the Hamiltonian operator, we use the states of a compact operator, which eventually helps to properly write the correlation functions, that have the proper desired functional form.

## INTRODUCTION

We start off by realizing the global conformal group of the 0+1 d case, which is SO(2,1). The generators of this group are - the time translation generator(Hamiltonian), the dilatation generator, and the special conformal generator.

The conformally invariant hamiltonian of this theory is -

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - g\phi^{\frac{2d}{d-2}} \tag{1}$$

Putting d=1, we represent the field as a single operator $q(t)$, which only depends on time. So write the lagrangian as -

$$L = \frac{1}{2}\dot{q}^2 - \frac{g}{2q^2} \equiv L(q,\dot{q}) \tag{2}$$

We know that the generators of the conformal group SO(2,1) follow the following algebra -

$$[H,D] = iD, \quad [K,D] = -iK, \quad [H,K] = 2iD \tag{3}$$

Let 'G' be any generator. In general it can be a linear combination of the three generators. This satisfies the relation -

$$\frac{\partial G}{\partial t} + i[H,G] = 0 \tag{4}$$

Using this equation, we can get the expression of generators, in terms of the field operators(q(t),$\dot{(q)}(t)$). Here we will see that the states of the hamiltonian, isn't discrete and non-normalizable. So, instead we diagonalize an compact operator, R to get a discrete spectrum, which is normalizable.

The vaccum state of this spectrum is not conformally invariant. The operator having some scaling dimension to define the correlation function, is also not of the primary nature and yet, we can find that, the 2-point correlation function is -

$$\langle O(x)O(y)\rangle \propto \frac{1}{|x-y|^{2\Delta}} \tag{5}$$

and 3-point is -

$$\langle O_1(w)O_2(x)O_3(y)\rangle \propto$$

$$\frac{1}{|w-x|^{\Delta_1+\Delta_2-\Delta_3}|x-y|^{\Delta_2+\Delta_3-\Delta_1}|y-w|^{\Delta_3+\Delta_1-\Delta_2}} \tag{6}$$

## PROPERTIES OF THE CONFORMAL GROUP

First, we start by writing down the lagrangian again -

$$L = \frac{1}{2}\dot{q}^2 - \frac{g}{2q^2}$$

The canonical momentum operator for this theory is -

$$p(t) = \frac{\partial L}{\partial \dot{q}} = \dot{q}(t) \tag{7}$$

We define the commutator for the quantum-mechanical treatment as - $[q(t),p(t)] = i$

Now, the Hamiltonian H, can be written as -

$$H = \frac{1}{2}p^2 + \frac{g}{2q^2} \equiv L(q,\dot{q}) \quad (g>0) \tag{8}$$

Then, the rest of the generators can be written as -

$$D = Ht - \frac{1}{4}(pq+qp), \quad K = Ht^2 - \frac{1}{2}t(pq+qp) + \frac{1}{2}q^2 \tag{9}$$

The procedure to compute the generators in q-basis is given in the appendix below.

In the Cartan basis, we write the generators and their algebra as -

$$R = \frac{1}{2}(\frac{K}{a} + aH), \;\; L_\pm = \frac{1}{2}(\frac{K}{a} - aH) \pm iD$$

$$\& \;\; [R, L_\pm] = \pm L_\pm, \;\; [L_-, L_+] = 2R \tag{10}$$

Here "a" is a parameter of the dimension of time. $R$ actually corresponds to the only compact subgroup of

SO(2,1) which is SO(2). $L_\pm$ is like the ladder operator we are used to seeing in angular momentum algebra. The Casimir is defined as -

$$C = R^2 - L_+L_- = \frac{1}{2}(HK + KH) - D^2 = \frac{g}{4} - \frac{3}{16} \quad (11)$$

H, D, and K are in general time-independent but, here we have a single operator q(t), which is a conformal primary of scale dimension $-1\backslash 2$. So, we can actually write the operators in a fashion, which is time dependent and satisfies these relations -

$$i[H, q(t)] = \frac{d}{dt}q(t) \quad (12)$$

$$i[D, q(t)] = t\frac{d}{dt}q(t) - \frac{1}{2}q(t) \quad (13)$$

$$i[k, q(t)] = t^2\frac{d}{dt}q(t) - tq(t) \quad (14)$$

## THE ISSUE AND ITS SOLUTION

We require the states and operators of any CFT to be able to reproduce the correct form of correlation functions. For that, we require the states to be conformally invariant under each of the group's generators.

The $SO(2,1)$ states are not invariant under the three transformations and also the states are not normalizable. Even the states, that we get from the compact operator can not be used directly to compute the correlation functions, cause those aren't fully conformally invariant.

Still, we can manage to find the correlation functions using the prescription, which is discussed below. First, we need to work with the states of the compact operator, R. We can define "$|n\rangle$" as -

$$R|n\rangle = r_n|n\rangle$$
$$r_n = r_0 + n, \quad n = 0, 1, ..., \quad r_0 > 0$$
$$\langle n'|n\rangle = \delta_{n',n} \quad (15)$$

The ladder operators are written as -

$$L_\pm|n\rangle = \sqrt{r_n(r_n \pm 1) - r_0(r_0 - 1)}|n \pm 1\rangle \quad (16)$$

So, in general, we can write any state "$|n\rangle$" as -

$$|n\rangle = \sqrt{\frac{\Gamma(2r_0)}{n!\Gamma(2r_0 + n)}}(\text{Ł}_+)^n|0\rangle \quad (17)$$

The operators are all constructed w.r.t to the phase space variable(q(t) and p(t)), which is what we do when doing 'quantum mechanics'. However, these variables are ultimately a function of time. So, it is wise to construct an " t " basis, which dAFF [2] . prescribed

We promote - $H$ to $i\frac{d}{dt}$ . We also demand that the form of the Casimir holds, which is -

$$C = \frac{1}{2}(HK + KH) - D^2 = r_0(r_0 - 1) = \frac{g}{4} - \frac{3}{16}$$
$$\therefore r_0 = \frac{1}{2}(1 + \sqrt{g + \frac{1}{4}}) \quad (18)$$

Using the commutation relation of the SO(2,1) group and the above relation, this leads us to the following representation -

$$D = i(t\frac{t}{dt} + r_0) \quad (19)$$

$$K = i(t^2\frac{d}{dt} + 2r_0 t) \quad (20)$$

'$-r_0$' looks like a scaling dimension in the expression of the given generators, which in turn depends upon the coupling constant g. These generators in the " t " basis satisfy the lie-algebra. We then compute the time-dependent functions $\beta_n(t)$ which are acted on by the generators on the said basis, first shown by dAFF [2] . Solving the equation -

$$\langle t|R|n\rangle = r_n\langle t|n\rangle$$
$$= \frac{1}{2}[(a + \frac{t^2}{a})\frac{d}{dt} + 2r_0\frac{t}{a}]\langle t|n\rangle \quad (21)$$

$$\langle t|n\rangle = \beta_n(t) = (-1)^n\left[\frac{\Gamma(2r_0 + n)}{n!}\right]^{1/2}\left(\frac{a - it}{a + it}\right)^{r_n}\frac{1}{(1 + \frac{t^2}{a^2})^{r_0}} \quad (22)$$

With these $\beta_n(t)$ functions, we can find the correlation functions. The two-point correlation function, as shown in [2] is -

$$\langle t_1|t_2\rangle = \sum_n \beta_n(t_1)\beta_n^*(t_2) \;=\; \frac{\Gamma(2r_0)}{[2i(t_1 - t_2)]^{2r_0}}$$

similarly, the 3-point correlation function is -

$$\langle t_1|B(t)|t_2\rangle \;\propto\; \frac{1}{|t - t_1|^\delta|t_2 - t|^\delta|t_1 - t_2|^{-\delta+2r_0}} \quad (23)$$

Here $B(t)$ is an unspecified primary field of scaling dimension $\delta$. From the relation above, we can construct the '$|t\rangle$' states as -

$$\sum_n |n\rangle\langle n|t\rangle = |t\rangle = \sum_n \beta_n^*(t)|n\rangle \quad (24)$$

The four-point functions can be built from the conformal blocks. Fixing any of the three points at 1,0 and $\infty$ and using cross ratios.

Using formulas (17) and (21) we construct the 't' states as -

$$|t\rangle = O(t)|0\rangle \quad (25)$$

Even though $|0\rangle$ isn't a conformally invariant vacuum, we'll see that along with $O(t)$ it does create the appropriate correlators, as if, $O(t)$ is a conformal primary with dimension $r_0$.

### THE CORRESPONDENCE BETWEEN NON-INVARIANT VACUUM AND NON-PRIMARY OPERATOR

For $d \geq 2$ , there exist some normalized vacuum $|\Omega\rangle$ such that -

$$H|\Omega\rangle = D|\Omega\rangle = K|\Omega\rangle = 0 \tag{26}$$

If there's some primary field $\mathcal{O}_\Delta(t)$ with scaling dimension $\Delta$ then -

$$\begin{aligned} i[H, \mathcal{O}_\Delta(0)] &= \dot{\mathcal{O}_\Delta}(0) \\ i[D, \mathcal{O}_\Delta(0)] &= \Delta\mathcal{O}_\Delta(0) \\ i[K, \mathcal{O}_\Delta(0)] &= 0 \end{aligned} \tag{27}$$

We will see that even if the operator, that we can make out of our theory, doesn't satisfy eq.(27) along with $|\Omega\rangle$.

This state is only annihilated by H. The new state is defined as $|\mathcal{O}_\Delta\rangle = \mathcal{O}_\Delta(0)|\Omega\rangle$.

Doing further calculations it is seen that -

$$e^{-Ha}|\mathcal{O}_{r_0}\rangle \propto |0\rangle \tag{28}$$

From this, the 2-point correlator $G_2(t_1 - t_2)$ boils down to this formula -

$$\begin{aligned} G_2(t_1 - t_2) &= \langle\Omega|\mathcal{O}_{r_0}(t_1)\mathcal{O}_{r_0}(t_2)|\Omega\rangle \\ &= \langle\Omega|\mathcal{O}_{r_0}(0)e^{-i(t_1-t_2)H}\mathcal{O}_{r_0}(0)|\Omega\rangle \\ &= \langle\Omega|\mathcal{O}_{r_0}(0)e^{-Ha}e^{[2a-i(t_1-t_2)]H}e^{-Ha}\mathcal{O}_{r_0}(0)|\Omega\rangle \\ &= \langle 0|e^{2a-i(t_1-t_2)H}|0\rangle \end{aligned} \tag{29}$$

The translational invariance is clear and now if we differentiate with respect to t, we get the following equation -

$$\left(t\frac{\partial}{\partial t} + 2r_0\right)G_2(t) = 0 \tag{30}$$

This has the solution - $G_2(t) \propto |t|^{-2r_0}$ Which is the desired solution and similarly we can determine the 3-point correlator.

### APPENDIX

In q basis, the form of the operator can be calculated from the eq.(4) like -

$$\frac{\partial D}{\partial t} + i[H, D] = 0 \implies D(t) = \int H dt = Ht + c$$

Using the commutation relation - $[H, D] = iH$, we get -

$$p[p, c] + [p, c]p + \frac{g}{q}[\frac{1}{q}, c] + [\frac{1}{q}, c]\frac{g}{q} = i(p^2 + \frac{g}{q^2})$$

$c \sim pq$. Because the operators are hermitian, we symmetrize the result, to get the desired results. Similarly, we can find the q-basis form of the K(SCT) operator.

The operator $O(t)$ in eq.(25) is calculated to be -

$$\begin{aligned} O(t) &= N(t)\ \exp(-\omega(t)L_+) \\ N(t) &= \Gamma(2r_0)^{1/2}\left(\frac{\omega(t)+1}{2}\right) \end{aligned}$$

where,

$$\omega(t) = \frac{a + it}{a - it} = e^{i\theta} \tag{31}$$

This can be viewed from the point of view of a projective transformation.

To come to the conclusion of eq.(28) this expression has to be evaluated -

$$Re^{-Ha}|\mathcal{O}_\Delta\rangle = e^{-Ha}(\frac{K}{2a} + iD)|\mathcal{O}_\Delta\rangle$$

$$= e^{-Ha}\,e^{-L_+}(R - \frac{1}{4}L_-)|\mathcal{O}_\Delta\rangle$$

$$= \Delta e^{-Ha}|\mathcal{O}_\Delta\rangle \tag{32}$$


[1] Claudio Chamon, Roman Jackiw, So-Young Pi, and Luiz Santos, "Conformal quantum mechanics as the CFT1 dual to AdS2," arXiv:1106.0726v1 [hep-th] 3 Jun 2011.

[2] V. de Alfaro, S. Fubini and G. Furlan, "Conformal Invariance in Quantum Mechanics," Nuovo Cim. 34A, 569 (1976).