Percolation as a Minimal Model, $c_{(6,1)} = -24$ CFT

Md Sariful Islam[1]

[1]*Department of Physics, Indian Institute of Technology - Kanpur, Kanpur, UP 208016*

In this review paper of this paper[1], important features of two-dimensional bond percolation on an infinite square lattice at its critical point within a conformal field theory (CFT) approach are presented. This approach is a level three null vector interpretation for Watts' differential equation describing the horizontal vertical crossing probability Π_{hv}. We will show that this differential equation can be derived from a level three null vector condition of a rational $c = -24$ CFT and see how this solution may be fitted into known properties of percolation.

INTRODUCTION

What is percolation?

In 1957, Broadbent and Hammersley were the first to formulate the percolation problem by asking the question of how probable it is for the center of a porous stone to be wet when laid into a jar of water. So "percolation" means a process of random walks through a material depending on the likelihood of ways to be opened or closed. Obviously, the probability depends on the size, shape, and number of open pores of the material.

It is usually modeled based on a lattice, e.g. a subset of $\mathbb{Z}^2$ (the plane square lattice) or the triangular lattice, whose bonds or sites are opened (or closed) with a probability p (or $(1-p)$), $p \in [0,1]$. In the following we will consider bond percolation on the square lattice, since the square lattice is dual to itself, it makes things much easier to calculate. Due to its finite size a stone may only be represented by a large but finite subset of $\mathbb{Z}^2$, but in physics, it is often easier to deal with infinitely large systems or with less dimensions. Thus, in our model, a vertex of the stone will be wet iff there exists a path in $\mathbb{Z}^2$ to some vertex at the boundary running through open bonds. This random subgraph obviously depends on the probability of the bonds being opened or closed and on the aspect ratio(r) of our two-dimensional rectangular stone. Solving the system numerically, it is found that there exists a critical probability for the bonds or sites to be opened. Suppose we have bonds connecting nearest neighboring sites with a probability p. Now let $\Pi_h(r)$ be the probability of having a cluster of open bonds spanning from left to right and thus establishing a horizontal crossing through the lattice. In the limit, the lattice sizes approach infinity, there exists a critical probability p_c such that $\Pi_h(r) = 0$ for $p < p_c$ and $\Pi_h(r) = 1$ for $p > p_c$. For $p_c = 1/2$ one may find that $\Pi_h(r) = 1/2$.

Application of Percolation

Percolation is useful as the $Q \to 1$ limit of the Q states Potts model or as a usage of $SLE(\kappa, \rho)$. It can also be used as a model for the conductivity of random resistance networks, the spreading of diseases, and forest fires. Another application is the error probability in wafer production.

A BRIEF REVIEW OF PERCOLATION PROPERTIES

According to Langlands et al[2], critical percolation in two dimensions has interesting features in conformal field theory such as the conformal invariance of the three independent crossing probabilities $1, \Pi_h, \Pi_{hv}$. As for Π_h, Cardy[3] has derived an exact solution using boundary conformal field theory which agrees with numerical data to a high accuracy. Motivated by this, Watts[4] tries to construct boundary operators for Π_{hv} in the context of the $Q \to 1$ limit of the Q states Potts model. He managed to derive one of the fifth-order differential equations that agrees with the simulation. Additionally, he observed that the three physically relevant solutions already satisfy a third-order differential equation.

In the previous literature, several arguments have been given to describe the crossing probabilities in two-dimensional critical percolation as conformal blocks of a four-point correlation function of $(h = 0)$-operators in a $c = 0$ conformal field theory, using a second (third) level null vector to get Π_h (Π_{hv}). The most prominent are

- (for $c = 0$) the Beraha numbers $Q = 4\cos^2\frac{\pi}{n}$ (with n usually denoted as $m+1 = 2, 3, 4...$ which in most Potts models are related to the central charge by $c = 1 - \frac{6}{m(m+1)}$
- (for $c = 0$) Π_h can be derived mathematically by the Stochastic/Schramm Loewner Evolution(SLE) which strengthens the first argument
- (for $h = 0$) the proportionality of the partition functions for free boundary condition to $Z = 1$, where Z is the partition function of percolation.
- (for $c = h = 0$) the interpretation of the central charge as describing the finite size effects of the energy which are believed to be absent.

BOND PERCOLATION AND THE Q-STATE POTTS MODEL[5]

Bond percolation can be explained by taking the $Q \to 1$ limit of the Q-state Potts model. In this model on each site of the lattice, there is a discrete variable σ_i(spin) that takes one of the Q possible values. So, the energy of the system is

$$E = J \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j} \tag{1}$$

It means that the same spin linking bonds have energy J and other bonds have zero energy. The partition function may be expressed as follows:

$$Z = \sum_{\langle \sigma \rangle} \prod_{\langle ij \rangle} (1 + \exp(-\beta J) \delta_{\sigma_i \sigma_j}) \tag{2}$$

This partition function can be realized in a different way similar to the percolation problem. Let p and (1-p) be the probability of activation and deactivation of each bond respectively. Also, each bond can be 'color' with Q possible values. So a given cluster of bonds is 'colored' according to the value of σ it supports. Then the partition function is given by

$$Z = \sum_{R} p^{B(R)} (1-p)^{B-B(R)} Q^{N_c(R)} \tag{3}$$

where B is the total number of bonds, B(R) is the total number of activated bonds, R is the subset of bonds that are activated, and $N_c(R)$ is the number of disjoint clusters in R. We have to also take clusters of size zero in order to have perfect correspondence with the Q-state Potts model. This correspondence allows us to formulate the problem of the crossing probability($\Pi_h(r)$) in terms of the partition function with different boundary conditions. Suppose $Z_{\alpha\beta}$ is the partition function on a rectangular lattice for the given boundary conditions- spins on the left side are in state α and on the right side are in state β and spins on the top and bottom sides are free. Then the crossing probability($\Pi_h(r)$) is given by

$$\Pi_h(r) = \lim_{Q \to 1} (Z_{\alpha\alpha} - Z_{\alpha\beta}) \tag{4}$$

Where $\alpha \neq \beta$. From boundary CFT we know that the partition functions of systems with boundary conditions can be given by the correlator of the boundary operators given at the points at which the conditions change. So the partition functions are given by
$Z_{\alpha\alpha} = Z_f \langle \phi_{f\alpha}(x_0) \phi_{\alpha f}(x_1) \phi_{f\alpha}(x_2) \phi_{\alpha f}(x_3) \rangle$
$Z_{\alpha\beta} = Z_f \langle \phi_{f\alpha}(x_0) \phi_{\alpha f}(x_1) \phi_{f\beta}(x_2) \phi_{\beta f}(x_3) \rangle$
Now Q-state Potts model is nothing but the minimal model $M(m, m-1)$ with $Q = 4\cos^2(\pi/m)(m = 3, 4, 6, \infty)$. It can be shown that the correct choice for the boundary operator for $\Pi_h(r)$ is $\phi_{\alpha f} = \phi_{(1,2)}$. Assuming scale invariance, $c = 0$ and using level two null vector condition for $\phi_{(1,2)}$ we get for $\Pi_h(r)$

$$\eta(1-\eta)g'' + \frac{2}{3}(1-2\eta)g' = 0 \tag{5}$$

It has two independent solutions, $\eta^{1/3}{}_2F_1(1/3, 2/3, 4/3; \eta)$ and 1. Taking into account correct asymptotic behavior we get

$$\Pi_h(r) = \frac{3\Gamma(2/3)}{\Gamma^2(1/3)} \eta^{1/3}{}_2F_1(1/3, 2/3, 4/3; \eta) \tag{6}$$

Next, we are going to show that this exact expression for horizontal crossing probability can be obtained using level three null vector condition on the boundary operators and using $c = -24$ instead of $c = 0$.

THE WATTS DIFFERENTIAL EQUATION

As mentioned before, Watts derived a fifth-order differential equation for Π_{hv}, using a $c = 0$ theory with $h_{(1,2)} = 0$ boundary changing operators following Cardy's ansatz for Π_h. In a $c = 0$ theory, Π_{hv} boundary operators cannot be identified directly contrary to Π_h. Considering the asymptotic behavior, one can find the correct expressions for Π_h and Π_h[6] by taking linear combinations of the three physically relevant solutions of

$$\frac{d^3}{dx^3}(x(x-1))^{4/3} \frac{d}{dx}(x(x-1))^{2/3} \frac{d}{dx} F(x) \tag{7}$$

where x is the crossing ratio and F is the conformally mapped crossing probability. The equation can be further factorized into

$$\left(\frac{d^2}{dx^2}(x(x-1)) + \frac{1}{2x-1}\frac{d}{dx}(2x-1)^2\right)\frac{d}{dx}(x(x-1))^{1/3} \frac{d}{dx}(x(x-1))^{2/3}\frac{d}{dx}F(x) \tag{8}$$

where the rightmost part gives us the three expected solutions for the crossing probabilities in percolation.

Interpretation as a level three null vector

If we simplify the third-order equation and compare it to the generic form of a level three null vector in the minimal model, it is found that there is no level three null vector in c=0 which could give rise to Watt's differential equation. Now, we will show that it can be derived from the null vector of an $h = h_{(1,3)} = \frac{-2}{3}$ field acting on a correlator containing $h_1 = h_2 = h_{(1,3)} = \frac{-2}{3}$ and $h_3 = h_{(1,5)} = -1$ in an $c_{(p,1)} = -24$ LCFT, which is a unique solution for level three null vector condition. According to [7], the level three null vector is given by

$$|\chi^{(3)}_{(h,c)}\rangle = (L^3_{-1} - 2(h+1)L_{-2}L_{-1} + h(h+1)L_{-3}|h\rangle \quad (9)$$

The differential operators $\mathcal{L}_{-n}$ are defined by

$$\mathcal{L}_{-n}(z) = \Sigma(\frac{(n-1)h_i}{(z_i - z)^n} - \frac{1}{(z_i - z)^{n-1}}\partial_{z_i}) \quad (10)$$

Letting them act on the four-point function $F(z, z_1, z_2, z_3) \equiv < \phi_h(z)\phi_{h_1}(z_1)\phi_{h_2}(z_2)\phi_{h_3}(z_3) >$ yields a quite lengthy expression. Replacing again all derivatives ∂_{z_i} by expressions only containing the derivative ∂_z and taking the limits $z_1 \to 0, z_2 \to 1 and\ z_3 \to \infty$, we get the third order differential equation for $F(z) \equiv F(z, 0, 1, \infty)$

$$\begin{aligned} 0 = \frac{d^3}{dz^3}F(z) + 2(h+1)\frac{2z-1}{z(z-1)}\frac{d^2}{dz^2}F(z) \\ +(h+1)(\frac{h-2h_1}{z^2} + \frac{h-2h_2}{(z-1)^2} - 2\frac{h_3 - h - h_1 - h_2}{z(z-1)} + \\ \frac{h}{z(z-1)})\frac{d}{dz}F(z) + h(h+1)(-\frac{2h_1}{z^2} - \frac{2h_2}{(z-1)^3} + \\ \frac{(2z-1)(h+h_1+h_2-h_3)}{z^2(z-1)^2})F(z) \end{aligned} \quad (11)$$

Next, we compare equation.(11) to Watt's differential equation in a suitable form

$$(\frac{d^3}{dz^3} + \frac{5(2z-1)}{z(z-1)}\frac{d^2}{dz^2} + \frac{4}{3z(z-1)}\frac{d}{dz})F(z) = 0 \quad (12)$$

Clearly, these equations can not be compared in this form. Using the generic form of the four-point function due to its conformal invariance $F(z)$ can be written as $F(z) = z^{\mu_{01}}(z-1)^{\mu_{02}}H(z)$ and inserting it in the equation.11 gives us a modified differential equation for $H(z)$ for which an appropriate choice of the h, h_1, h_2, h_3 is possible and that is $h = h_1 = h_2 = -2/3$ and $h_3 = -1$. This means that all four weights can be chosen from the Kac-table of one and the same minimal CFT. The equation belongs to $c_{(6,1)} = -24$ because the highest weight representation (-2/3) has indeed a third-level null vector.

Holding on to $c = 0$ in a tensor model

As shown above, the differential equation that gives us the correct solution for the Π_{hv} directs towards a $c = -24$ LCFT. But if we want to stick to $c = 0$ CFT, we may do so via giving a tensor ansatz of two CFTs, one of them being $c = -24$ as needed to satisfy Watt's differential equation and the other being $c = 24$. In this scenario, any correlation function or any field factorizes into two parts belonging to two CFTs respectively, i.e. $\Phi_h(z) = \Phi_{h,c=-24}(z) \otimes \Phi_{H-h,c=+24}(z)$. We also assume that the second factor of the third-level differential equation is

$$G_{c=+24}(z) = \langle\Phi_h(z)\Phi_{h_1}(0)\Phi_{h_2}(1)\Phi_{h_3}(\infty)\rangle_{c=+24} \quad (13)$$

But all information is contained in the first factor given by

$$F_{c=-24}(z) = \langle\Phi_{-2/3}(z)\Phi_{-2/3}(0)\Phi_{-2/3}(1)\Phi_{-1}(\infty)\rangle_{c=-24} \quad (14)$$

A perfect match would be to find

$$H(z) = F_{c=-24}(z)G_{c=+24}(z) \Longrightarrow G_{c=+24}(z) = z^{-1/3}(z-1)^{-1/3} \quad (15)$$

A possible solution will be if $G(z)$ is a three-point function, i.e. $\langle\Phi_{1/3}(z)\Phi_{1/3}(0)\Phi_{1/3}(1)\mathbb{I}(\infty)\rangle_{c=+24}$ It remains to clarify whether such a correlator exists and is non-vanishing in a $c = +24$ theory.

Cardy's formula and $c = -24$

We see that there are two differential equations numerically 'proven' to be correct but coming from two different CFTs, both assumed to describe percolation. So the natural question will be which one is correct? Is there an interpretation of Cardy's formula[3] for Π_h in $c = -24$? In general, Cardy's formula arises from a level two null vector condition applied to a four-point correlation function,

$$\begin{aligned} (\frac{3}{2(2h+1)}\frac{d^2}{dz^2} + \frac{2z-1}{z(z-1)}\frac{d}{dz} - \frac{h_1}{z^2} - \frac{h_2}{(z-1)^2} \\ + \frac{h+h_1+h_2-h_3}{z(z-1)})F(z) = 0 \end{aligned} \quad (16)$$

For $c = -24$, we have $\phi_{(1,2)}$ with weight $h = h_{(1,2)} = -3/8$. Now we have to check whether the solutions of $c = -24$ span the solution space as those for $c = 0$ since the latter has been proven to be correct by the numerical simulation of Langlands et. al.[2]. Hence $F(z)$ should be of the form ${}_2F_1(1/3, 2/3, 4/3, z)$. From the above equation it can be shown $h_1 = h_2 = h_3 = h_{(1,4)} = -\frac{7}{8}$ and $F_1(z) = (z(z-1))^{1/4}z^{1/3}{}_2F_1(1/3, 2/3, 4/3, z)$ and another solution will be $F_2(z) = (z(z-1))^{1/4}$. So, by comparing with Cardy crossing probability for percolation is given by their quotient $\Pi_h \propto F_1/F_2$. Our solution has same asymptotic behavior, i.e. vanishes for $z \to 1$ and goes to one for $z \to 0$. The normalization is obtained by considering the identity

$$\begin{aligned} \frac{3\Gamma(2/3)}{\Gamma^2(1/3)}{}_2F_1(1/3, 2/3, 4/3, z) = 1 - \frac{3\Gamma(2/3)}{\Gamma^2(1/3)}(1-z)^{1/3} \\ {}_2F_1(1/3, 2/3, 4/3, 1-z) \end{aligned} \quad (17)$$

which gives $\frac{3\Gamma(2/3)}{\Gamma^2(1/3)}$ as a normalization factor. This result is remarkable, since it contains the two fields for critical exponents in percolation, i.e. $h_{(1,2)} = -3/8$ and $h_{(1,4)} = -7/8$ in the $c = -24$ theory.

CONCLUSION

In this review paper, we have shown that if we want to describe two-dimensional bond percolation within a conformal field theory, using a level three null vector condition to get a differential equation for horizontal-vertical crossing probability Π_{hv} that fits the numerical data, we have to take $c = -24$. This solution is unique. Also, there are no strict arguments contradicting our result, even not from the derivation of the horizontal crossing probability Π_h whose form has already been proven in the literature, since it can be explained in our $c = -24$ CFT proposal as well. Hence the question remains if we should consider percolation being rather a $c = -24$ than the commonly assumed $c = 0$ theory of percolation.

[1] M. Flohr and A. Müller-Lohmann, Journal of Statistical Mechanics: Theory and Experiment **2005**, P12004 (2005), URL https://doi.org/10.1088%2F1742-5468%2F2005%2F12%2Fp12004.

[2] R. Langlands, P. Pouliot, and Y. Saint-Aubin, *Conformal invariance in two-dimensional percolation* (1994), math/9401222.

[3] J. L. Cardy, Journal of Physics A: Mathematical and General **25**, L201 (1992), URL https://doi.org/10.1088%2F0305-4470%2F25%2F4%2F009.

[4] G. M. T. Watts, Journal of Physics A: Mathematical and General **29**, L363 (1996), URL https://doi.org/10.1088%2F0305-4470%2F29%2F14%2F002.

[5] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics (Springer-Verlag, New York, 1997), ISBN 978-0-387-94785-3, 978-1-4612-7475-9.

[6] P. Kleban and D. Zagier, Journal of Statistical Physics **113**, 431 (2003), URL https://doi.org/10.1023%2Fa%3A1026012600583.

[7] S. Moghimi-Araghi, S. Rouhani, and M. Saadat, Nuclear Physics B **599**, 531 (2001), URL https://doi.org/10.1016%2Fs0550-3213%2801%2900004-9.