Scale vs Conformal Invariance

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This article is a project report on scale vs conformal invariance, which briefly summarizes their distinction and possible equivalence. It aims to understand how a scale invariant theory implies conformal invariance. To study these characteristics, it starts by formulating the structure of the energy momentum tensor. Specifically, it tries to emphasize how the trace of this tensor behaves in various theories and backgrounds by describing its relevant anomalies and symmetric properties. Finally, at the end, it provides proof of the enhancement of conformal invariance from the scale invariance.

I. INTRODUCTION

This article discusses the problem of scale vs conformal invariance in relativistic quantum field theories. It distinctly clarifies the conditions under which a given scale invariant field theory has the enhanced conformal symmetry. It emphasizes the trace of energy momentum tensor captures the condition through a local operator called Virial current. Further, it provides strategy to show the enhancement of conformal invariance from scale invariance under which the theory is considered. The improvement of energy momentum tensor also takes account of other suitable theorems and renormalization group effects, which we have to consider in order to understand the complete picture of enhanced conformal invariance in any arbitrary dimension. Therefore, it is crucial to understand how to compute the trace of energy momentum tensor. Its significance to the renormalization scheme is thoroughly presented in [1].

II. SCALE AND CONFORMAL INVARIANCE IN QFT

In QFTs, the Poincare invariance give rise to the following algebra on the background of spacetime symmetry.

$$\begin{aligned} i[J^{\mu\nu}, J^{\rho\sigma}] &= \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu} \\ i[P^{\mu}, J^{\rho\sigma}] &= \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho} \\ [P^{\mu}, P^{\nu}] &= 0 \end{aligned} \tag{1}$$

In this Poincare transformation, one can further extend the symmetry to a Dilatation current using the generator D as

$$\begin{aligned} i[P^{\mu}, D] &= -P^{\mu} \\ [J^{\mu\nu}, D] &= 0 \end{aligned} \tag{2}$$

Further, a full symmetry group can be enhanced from here by augmenting K^{μ}; a special conformal transformation

$$\begin{aligned} [K^{\mu}, D] &= -iK^{\mu} \\ [P^{\mu}, K^{\nu}] &= 2i\eta^{\mu\nu} D + 2iJ^{\mu\nu} \\ [K^{\mu}, K^{\nu}] &= 0 \\ [J^{\rho\sigma}, K^{\mu}] &= i\eta^{\mu\rho} K^{\sigma} - i\eta^{\mu\sigma} K^{\rho} \end{aligned} \tag{3}$$

This closure structure of this group suggests that conformal invariance indeed implies scale invariance. However, the converse of this situation is also possible in some theories which will be briefly discussed in the coming sections.

One of the important way to make a distinction in these symmetries is by closely studying the energy-momentum tensor. We know by Noether prescription spacetime symmetries leads to conserved em tensor; $\partial^{\mu} T_{\mu\nu} = 0$. For a scale invariant theory, it requires to satisfy that,

$$T^{\mu}_{\mu} = \partial^{\mu} J_{\mu} \qquad \text{when } x^{\mu} \to \lambda x^{\mu} \tag{4}$$

whose corresponding conserved scale current is $D_{\mu} = x^{\rho} T_{\mu\rho} - J_{\mu}$. Here, J_{μ} is known as the Virial current [2]. For a conformal invariant theory, it requires;

$$T^{\mu}_{\mu} = 0 \qquad \text{when } x^{\mu} \to \frac{x^{\mu} + v^{\mu} x^2}{1 + 2v^{\mu} x_{\mu} + v^2 x^2} \tag{5}$$

and the corresponding special conformal current is $K_{\mu} = \left[\rho_{\nu} x^2 - 2x_{\nu}(\rho_{\sigma} x^{\sigma})\right] T^{\nu}_{\mu}$. However, this will not always the case, because the em tensor are not unique in general. This non-uniqueness leads to important consequences in conformal invariance. This can be avoided by improving the em tensor by employing the method similar to construction of Belinfante tensor [3]. Suppose, the trace of em tensor appears in the form as [Appendix-A]:

$$\begin{aligned} T^{\mu}_{\mu} &= \partial^{\mu} \partial^{\nu} L_{\mu\nu} \quad (\text{for } d \geq 3), \\ T^{\mu}_{\mu} &= \partial^{\mu} \partial_{\mu} L \quad (\text{for } d = 2) \end{aligned} \tag{6}$$

Where L and $L_{\mu\nu}$ are local operators and the improvement on these tensors will lead to the respective traceless and symmetric tensors [Appendix-B] given that the conservation of $T_{\mu\nu}$ is still preserved [4].

$$\begin{aligned} \Theta_{\mu\nu} &= T_{\mu\nu} + \frac{1}{d-2}(\partial_{\mu}\partial_{\alpha} L^{\alpha}_{\nu} + \partial_{\nu}\partial_{\alpha} L^{\alpha}_{\mu} \\ &\quad -\partial^2 L_{\mu\nu} - \eta_{\mu\nu}\partial_{\alpha}\partial_{\beta} L^{\alpha\beta}) \\ &\quad + \frac{1}{(d-2)(d-1)}\left(\eta_{\mu\nu}\partial^2 L - \partial_{\mu}\partial_{\nu} L\right) \quad (\text{for } d \geq 3) \\ \& \ \Theta_{\mu\nu} &= T_{\mu\nu} + \left(\eta_{\mu\nu}\partial^2 L - \partial_{\mu}\partial_{\nu} L\right) \qquad (\text{for } d = 2) \end{aligned} \tag{7}$$

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III. WEYL ANOMALY AND CURVED BACKGROUND

We have argued that the tracelessness condition of em tensor is governed from the conformal invariance. This condition easily yields from Weyl invariance of a flat metric. However, if we invoke the CFT on a curved background, this will result into non-vanishing trace, known as Weyl anomaly [5, 7]. In $2d$ we can easily show how this anomaly appears, by letting the curved background varying infinitesimally close to the flat space. So then considering the infinitesimal Weyl transformation as $\delta g_{\alpha\beta} = 2\omega\delta_{\alpha\beta}$, consequently will lead to the following variation in em tensor as briefly shown in [Appendix-C].

$$\langle T^{\mu}_{\mu} \rangle = -\frac{c}{12} R \tag{8}$$

where the scalar curvature $R = -2\partial^2\omega$ and c is the central charge which arises from the OPE calculations.

Similarly, in $d = 4$ dimension, the most generic possibility of Weyl anomaly is [5, 6]

$$\begin{aligned} \langle T^{\mu}_{\mu} \rangle &= cC^2 - aE + bR^2 + \tilde{b} D^{\mu}_{\mu} R \\ &\quad + d\epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}_{\mu\nu} + d\epsilon^{\mu\nu\rho\sigma} R^{\alpha\beta}_{\mu\nu} R_{\alpha\beta\rho\sigma} \end{aligned} \tag{9}$$

where the Weyl tensor C is expressed as $C^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ and the Euler scalar is $E = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$.

Now, for a Weyl inavriant theory, the important thing it ensures that the metric of flat space still remains invariant by an overall Weyl scalar. This induces the diffeomorphism as

$$ds^2 = \Omega(\tilde{x})\eta_{\mu\nu} d\tilde{x}^{\mu} d\tilde{x}^{\nu} = \eta_{\mu\nu} dx^{\mu} dx^{\nu} \tag{10}$$

Thus, the em tensor need not to be unique for a given CFT. We can still able to couple a CFT to gravity in a non-Weyl invariant way. Also, we can obtain the required conformal invariance structure by remaining on this background. So, covariantly, we can employ the curved space prescription to the problem of scale vs conformal invariance. Suppose, the action is scale invariant (i.e. $g_{\mu\nu} \to e^{2\bar{\sigma}} g_{\mu\nu}$), where $\bar{\sigma}$ is a spacetime independent constant and the action density is scale invariant up to a total derivative term $\delta L = -\bar{\sigma} D^{\mu} J_{\mu}$

$$T^{\mu}_{\mu} = \frac{2}{\sqrt{|g|}} g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} = D^{\mu} J_{\mu} \tag{11}$$

This is the origin of Virial current from the viewpoin of curved background. While if the action is Weyl invariant, as $g_{\mu\nu} \to e^{2\sigma(x)} g_{\mu\nu}$, where $\sigma(x)$ is spacetime dependent arbitrary scalar function, then the em tensor is traceless.

$$T^{\mu}_{\mu} = \frac{2}{\sqrt{|g|}} g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} = 0 \tag{12}$$

IV. SOME EXAMPLES

A. Free Massless Scalar Theory

In d dimension, the action of such system which is minimally coupled with gravity is given as:

$$S = \frac{1}{2} \int d^d x \sqrt{|g|} (\partial^{\mu}\phi \partial_{\mu}\phi) \tag{13}$$

The canonical em tensor is

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}} \bigg|_{g_{\mu\nu} = \eta_{\mu\nu}} = \partial_{\mu}\phi \partial_{\nu}\phi - \frac{\eta_{\mu\nu}}{2} (\partial_{\rho}\phi)^2 \tag{14}$$

The trace will then follow as

$$T^{\mu}_{\mu} = \frac{2-d}{2} (\partial_{\mu}\phi)^2 = \frac{2-d}{4} (\Box\phi^2) \tag{15}$$

This gives us the scale invariant free massless scalar theory, whose Virial current is given by

$$J_{\mu} = \frac{2-d}{2} \phi \partial_{\mu}\phi \tag{16}$$

Here, the theory can be also conformal invariant in any dimension as the form of trace is $T^{\mu}_{\mu} = \partial^{\mu}\partial^{\nu} L_{\mu\nu}$, where

$$L_{\mu\nu} = \frac{2-d}{4} \eta_{\mu\nu} \phi^2 \tag{17}$$

B. Free Massless Dirac Theory

The em tensor can be studied in the exact same way and it will take the form as:

$$T_{\mu\nu} = i\frac{1}{2} \bar{\psi} (\gamma_{\mu}\partial_{\nu} + \gamma_{\nu}\partial_{\mu})\psi - i\eta_{\mu\nu} \bar{\psi} \gamma^{\rho} \partial_{\rho}\psi \tag{18}$$

Unlike free scalar theory, here the traceless feature exist in all dimension, thus the massless free fermion is conformal invariant in any dimension.

C. Free Maxwell Theory

The action for this free $U(1)$ theory in d dimension and its corresponding canonical gauge invariant em tensor are given as

$$S = \frac{1}{2} \int d^d x \sqrt{|g|} \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{19}$$

$$T_{\mu\nu} = F_{\mu\rho} F^{\rho}_{\nu} - \frac{\eta_{\mu\nu}}{4} (F_{\rho\sigma})^2 \tag{20}$$

The trace of this em tensor do not vanish when $d \neq 4$, because

$$T^{\mu}_{\mu} = \frac{4-d}{4} (F_{\rho\sigma})^2 = \frac{4-d}{8} \partial_{\mu} (A_{\rho} F^{\mu\rho}) \tag{21}$$

However, it is still a divergence of a current. Therefore, this theory is scale invariant with virial current:

$$J_{\mu} = \frac{4-d}{8} A^{\nu} F_{\mu\nu} \tag{22}$$

Therefore, for dimensions $d \neq 4$, the free Maxwell theory is only scale invariant and not conformal invariant, but in $d = 4$, it is conformal invariant.

V. PROOF OF ENHANCEMENT OF INVARIANCE

To understand the possible proof of enhancement from scale to conformal invariance, we can restrict to the well established proof in $d = 2$ dimension. Let us consider the two-point function of the em tensor $T_{\mu\nu}$ in complex coordinates. We define $T \equiv T_{zz}$ and $\Theta \equiv T^{\mu}_{\mu}$. Following Zamolodchikov [8], we can define:

$$\begin{aligned} F(|z|^2) &= z^4\langle T(z,\bar{z})T(0)\rangle \\ G(|z|^2) &= z^3\bar{z}\langle T(z,\bar{z})\Theta(0)\rangle \\ H(|z|^2) &= z^2\bar{z}^2\langle \Theta(z,\bar{z})\Theta(0)\rangle \end{aligned} \quad (23)$$

From Poincare invariance, we know that $T_{\mu\nu}$ is conserved

$$\bar{\partial}T + 4\partial\Theta = 0 \quad (24)$$

Now by taking the correlation function between this equation of motion and either T or Θ, one can derive the equations

$$\dot{F} + \tfrac{1}{4}(\dot{G} - 3G) = 0 \ \ \& \ \ \dot{G} - G + \tfrac{1}{4}(\dot{H} - 2H) = 0 \quad (25)$$
$$\text{where } \dot{X} \equiv z\bar{z}X'(z,\bar{z})$$

In a theory with coupling constants g_i, we can write the renormalization group equation for a function C, defined as $C \equiv 2F - G - \frac{3}{8}H$;

$$\left(R\frac{\partial}{\partial R} + \beta_i(g)\frac{\partial}{\partial g_i}\right)C(g,R) = 0 \quad (26)$$

where $R \equiv \sqrt{z\bar{z}}$ and β_i are the renormalization group beta functions. Also, using the equations of motion in (25), we can show $\dot{C} = -\frac{3}{4}H$.

According to Zamolodchikov's C-theorem, if renormalization flows connect different conformal field theories, then C decreases from the ultraviolet to the infrared with C equals to the central charge c at criticality. So, at a scale-invariant fixed point, we can assume the stress-energy tensor scales canonically so that $T_{\mu\nu}$ has a scaling dimension $\Delta = 2$ and C is constant. This follows [9]

$$\langle \Theta(z,\bar{z})\Theta(0)\rangle = 0 \quad (27)$$

Since, Θ is the trace of the em tensor, hence the scale invariance implies conformal invariance. Similarly in $d = 4$ dimension, from the analysis of Local renormalization group, it can be perturbatively shown how a-theorem holds true, such that scale invariance implies conformal invariance.

VI. CONCLUSION

Finally, this article attempted a very brief demonstration of how scale and conformal invariance arises in QFTs. With the help of some examples, it has conveyed how scale invariance indeed leads to conformal invariance. But, when it does not, it could be due to the possible inconsistent with some important assumptions of QFT. This is well argued in [1] with more effective examples. There is always a deep spacetime structure behind the enhancement of conformal invariance from scale invariance. As we see Zamolodchikov c-theorem in $2d$, its higher dimensional analogues can also be presented which would play a significant role in understanding the enhancement.

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Appendix A: Trace of stress tensor and Virial current

The infinitesimal scale transformation is given by $\delta x^{\mu} = \epsilon x^{\mu}$.

Following the Noether procedure, the dilatation current S^{μ} corresponding to scale invariance is

$$S^{\mu} = x^{\nu}T^{\mu}_{\nu}(x) + K^{\mu}(x) \quad (A1)$$

where $T^{\mu}_{\nu}(x)$ is the canonical em tensor and K^{μ} is the local Virial operator, so the conservation of scale current ensures that

$$T^{\mu}_{\mu} = -\partial_{\mu}K^{\mu} \quad (A2)$$

For infinitesimal conformal transformation, we have $\delta x^{\mu} = \epsilon b^{\mu}(x)$, such that

$$\partial_{\mu}b_{\nu}(x) + \partial_{\nu}b_{\mu}(x) = \frac{2}{d}g_{\mu\nu}\partial.b(x) \quad (A3)$$

Similarly, from Noether's prescription, we can calculate the following current

$$j^{\mu}_{b}(x) = b^{\nu}(x)T^{\mu}_{\nu}(x) + \partial.b(x)K'^{\mu} + \partial_{\nu}\partial.b(x)L^{\nu\mu}(x) \quad (A4)$$

Again, from the conservation of conformal current, we have

$$T^{\mu}_{\mu} = -\partial_{\mu}K'^{\mu}, \qquad K'^{\mu} = -\partial_{\nu}L^{\nu\mu}(x) \quad (A5)$$

This leads to the conditions on the trace of em tensor as given in (6).

Appendix B: Improved stress tensor

In defining the new energy momentum tensor, we have to ensure that it is still conserved and symmetric. The weakest requirement which was given by [2], to improve this tensor is by a fourth rank tensor field $C^{\lambda\rho\mu\nu}(x)$, similar to Belinfante tensor.

$$\Theta^{\mu\nu} = T^{\mu\nu} + \tfrac{1}{2}\partial_{\lambda}\partial_{\rho}C^{\lambda\rho\mu\nu} \quad (B1)$$
$$\text{where } C^{\lambda\rho\mu\nu} = -C^{\lambda\mu\rho\nu} = -C^{\nu\rho\mu\lambda} \quad (B2)$$
$$\& \ \partial_{\lambda}\partial_{\rho}C^{\lambda\rho\mu\nu} = \partial_{\lambda}\partial_{\rho}C^{\lambda\rho\nu\mu} \quad (B3)$$

For $d=4$, the fourth rank tensor field defined as:

$$\begin{aligned} C^{\lambda\rho\mu\nu} &= \eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\rho\nu}\sigma_+^{\lambda\mu} + \eta^{\mu\nu}\sigma_+^{\lambda\rho} \\ &\quad -\frac{1}{3}\eta^{\lambda\rho}\eta^{\mu\nu}\sigma_{+\alpha}^{\alpha} + \frac{1}{3}\eta^{\lambda\mu}\eta^{\rho\nu}\sigma_{+\alpha}^{\alpha} \end{aligned} \quad \text{(B4)}$$

Here, $\sigma_+^{\mu\nu}$ is similar to local operator $L^{\mu\nu}$ of equation (6), and it is related to the Virial current (in equation (4)), as $J^\mu = \partial_\nu\sigma^{\mu\nu}$. However, we need the symmetric part of this operator which is denoted as $\sigma_+^{\mu\nu}$.

For an arbitrary d dimension, this is given by;

$$\begin{aligned} C^{\lambda\rho\mu\nu} &= A\left(\eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\rho\nu}\sigma_+^{\lambda\mu} + \eta^{\mu\nu}\sigma_+^{\lambda\rho}\right) \\ &\quad +B\left(\eta^{\lambda\mu}\eta^{\rho\nu}\sigma_{+\alpha}^{\alpha} - \eta^{\lambda\rho}\eta^{\mu\nu}\sigma_{+\alpha}^{\alpha}\right) \end{aligned} \quad \text{(B5)}$$

where, $A = \frac{2}{d-2}$ and $B = \frac{A}{d-1} = \frac{2}{(d-2)(d-1)}$.

This leads to the final expression as;

$$\begin{aligned} \partial_\lambda\partial_\rho C^{\lambda\rho\mu\nu} &= A\left(\partial^2\sigma_+^{\mu\nu} - \eta^{\mu\nu}\partial_\lambda\partial_\rho\sigma_+^{\lambda\rho} - \partial^\mu\partial_\rho\sigma_+^{\rho\nu} - \partial^\nu\partial_\rho\sigma_+^{\rho\mu}\right) \\ &\quad +B\left(\partial^\mu\partial^\nu\sigma_{+\alpha}^{\alpha} - \eta^{\mu\nu}\partial^2\sigma_{+\alpha}^{\alpha}\right) \end{aligned} \quad \text{(B6)}$$

Using the properties of four field tensor, it can be proved that there are $\frac{d^2(d-1)^2}{4}$ independent components. In $d=2$ dimension, C^{0101} is the only unique non-zero independent component, thus in $d=2$, the stress tensor becomes

$$\Theta^{\mu\nu} = T^{\mu\nu} - \frac{1}{2}\epsilon^{\mu\lambda}\epsilon^{\nu\rho}\partial_\lambda\partial\rho C^{0101} \quad \text{(B7)}$$

where $\epsilon^{\mu\nu}$ is anti-symmetric Levi-Civita pseudo tensor with $\epsilon^{01} = +1$ and $C^{0101}(x)$ is a scalar field.

Appendix C: Weyl Trace Anomaly

starting from the conservation of em tensor, we have

$$\partial T_{z\bar{z}} = -\bar{\partial}T_{zz} \quad \text{(C1)}$$

Using OPE, we can write,

$$\begin{aligned} \partial_z T_{z\bar{z}}\partial_w T_{w\bar{w}} &= \bar{\partial}_{\bar{z}}T_{zz}\bar{\partial}_{\bar{w}}T_{ww} \\ &= \bar{\partial}_{\bar{z}}\bar{\partial}_{\bar{w}}\left(\frac{c/2}{(z-w)^4} + ...\right) \end{aligned} \quad \text{(C2)}$$

Further, we can also write;

$$\begin{aligned} \bar{\partial}_{\bar{z}}\bar{\partial}_{\bar{w}}\frac{1}{(z-w)^4} &= \frac{1}{6}\bar{\partial}_{\bar{z}}\bar{\partial}_{\bar{w}}\left(\partial_z^2\partial_w\frac{1}{z-w}\right) \\ &= \frac{\pi}{3}\partial_z^2\partial_w\bar{\partial}_{\bar{w}}\delta^{(2)}(z-w) \end{aligned} \quad \text{(C3)}$$

$$\Longrightarrow T_{z\bar{z}}(z,\bar{z})T_{w\bar{w}}(,\bar{w}) = \frac{\pi c}{6}\partial_z\bar{\partial}_{\bar{z}}\delta^{(2)}(z-w) \quad \text{(C4)}$$

Now, we can compute the variation in the trace of em tensor w.r.t. a small shift in the metric, i.e., some infinitesimally curved background close to the flat space.

$$\begin{aligned} \delta\langle T_\mu^\mu(\sigma)\rangle &= \delta\int D\phi e^{-S}T_\mu^\mu(\sigma) \\ &= \tfrac{1}{4\pi}\int D\phi e^{-S}\left(T_\mu^\mu(\sigma)\int d^2\sigma'\sqrt{g}\delta g^{\alpha\beta}T_{\alpha\beta}(\sigma')\right) \\ &\quad (\because\ \delta g^{\alpha\beta} = -2\omega\delta^{\alpha\beta},\ \text{for Weyl transformaton}) \\ &= -\tfrac{1}{2\pi}\int D\phi e^{-S}\left(T_\mu^\mu(\sigma)\int d^2\sigma'\omega(\sigma')T_\nu^\nu(\sigma')\right) \end{aligned} \quad \text{(C5)}$$

To compute the Weyl anomaly, we change the coordinates:

$$T_\mu^\mu(\sigma)T_\nu^\nu(\sigma') = 16T_{z\bar{z}}(z,\bar{z})T_{w\bar{w}}(w,\bar{w}) \quad \text{(C6)}$$

Also using the fact; $8\partial_z\bar{\partial}_{\bar{w}}\delta^{(2)}(z-w) = -\partial^2\delta^{(2)}(\sigma-\sigma')$. Substituting these we get,

$$T_\mu^\mu(\sigma)T_\nu^\nu(\sigma') = -\frac{c\pi}{3}\partial^2\delta(\sigma-\sigma') \quad \text{(C7)}$$

Then plugging this into the expression for$\delta\langle T_\mu^\mu(\sigma)\rangle$ and integrating by parts, we are left with

$$\delta\langle T_\mu^\mu(\sigma)\rangle = \frac{c}{6}\partial^2\omega \quad \text{(C8)}$$

Finally, we use the fact that we are working infinitesimally to replace $e^{-2\omega} = 1$, so that $R = -2\partial^2\omega$. Then

$$\langle T_\mu^\mu(\sigma)\rangle = -\frac{c}{12}R \quad \text{(C9)}$$

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