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## Truncated Conformal Space Approach in 2-d

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### INTRODUCTION

Truncated Conformal Space Approach (TCSA) is a Hamiltonian truncation method originally introduced by Zamolodchikov and Yurov [1]. This is a very useful tool to study some deformed CFT where conformal perturbation theory could not be directly applied due to strong coupling. This is a non-perturbative method in the sense that it does not calculate the correction of energy as a power series in the coupling. In many cases (especially for the minimal models) the full Hilbert space of some CFTs are known in terms of the primaries and their descendants. If a 2-d CFT is deformed by a relevant operator (conformal dimension  $< 2$ ), the spectrum of the deformed CFT can be obtained by using TCSA. The spectrum and the eigenstates of the deformed CFT can be found after some matrix diagonalisation. We can extract other information such as entanglement entropy, and we can also study the CFT in IR fixed point using TCSA, starting from a UV CFT. Here one example is checked.

In this project we will like to use TCSA (Truncated Conformal Space Approach) to find the non-perturbative truncated spectrum of non-integrable deformations of the Sine-Gordon model. First we shall review the model and its spectrum in the integrable limit. We will review the TCSA and how it reproduces the integrable limit.

### THE METHOD

To apply TCSA, we have to know the spectrum of the initial CFT, along with the different OPE coefficients of the primaries of the CFT. If a 2-d CFT has a set of primary operators given by  $\{(h_i, \bar{h}_i)\}$ , the Full Hilbert

space of the CFT can be written as a direct sum,

$$\mathcal{H} = \oplus_i L_{-n_1} L_{-n_2} \dots L_{-n_p} \bar{L}_{-m_1} \dots \bar{L}_{-m_q} |h_i, \bar{h}_i\rangle \quad (1)$$

Where the state  $|h_i, \bar{h}_i\rangle$  is created by a primary operator  $\Phi_{h_i, \bar{h}_i}$  acting on vacuum at infinite past.

$L_{-n_a}$  and  $\bar{L}_{-m_b}$  are the Virasoro generators corresponding to the holomorphic and antiholomorphic. The state  $L_{-n_1} L_{-n_2} \dots L_{-n_p} \bar{L}_{-m_1} \dots \bar{L}_{-m_q} |h_i, \bar{h}_i\rangle$  is called a descendant with conformal dimensions  $(h_i + n_1 \dots + n_p, (\bar{h}_i + m_1 \dots + m_q))$ .

The descendants are neither orthogonal nor normalized. The descendants form a overcomplete basis set, due to the presence of null states for some particular primary operators at some particular levels. The states corresponding to different levels are orthogonal. But different states at same level might not be orthogonal. A orthogonal basis can be formed in such cases by Gram-Schmidt procedure, which is diagonalisation of Kac matrix at that level. The unperturbed CFT Hamiltonian is already diagonal in this basis.

The deformed CFT Hamiltonian can be given by  $H = H_0 + V$ , now the operator  $V$  can be expressed in terms of primary and their descendants. The basis states can also be written as the linear combinations of primaries and their descendants. Thus the matrix elements of  $V$  and the full Hamiltonian  $H$  are some 3 point functions of CFT. If the OPE coefficients  $(\mathcal{C}_{\phi_1, \phi_2, \phi_3})$  are known, then to get the complete spectrum the full Hamiltonian matrix is calculated and diagonalised.

$$\langle \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) \rangle = \frac{\mathcal{C}_{\phi_1, \phi_2, \phi_3}}{(z_{12})^{h_1+h_2-h_3} (z_{23})^{h_2+h_3-h_1} (z_{13})^{h_3+h_1-h_2}} \times \frac{1}{(\bar{z}_{12})^{\bar{h}_1+\bar{h}_2-\bar{h}_3} (\bar{z}_{23})^{\bar{h}_2+\bar{h}_3-\bar{h}_1} (\bar{z}_{13})^{\bar{h}_3+\bar{h}_1-\bar{h}_2}} \quad (3)$$

### THE SINE-GORDON MODEL

The lagrangian density for the Sine-Gordon model can be given as

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - (1 - \cos \phi) \quad (4)$$

The potential  $U(\phi) = (1 - \cos \phi)$  is a periodic function of the field variable  $\phi$ . The Euler-Lagrange equation of motion is,

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \sin \phi = 0 \quad (5)$$

The above equation(5) is very similar to the Klein Gordon equation of motion except the  $\sin \phi$  term, and hence

the name Sine-Gordon model. The canonical stress tensor is,

$$\begin{aligned} T_{\mu\nu} &= \left( \frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi)} \right) \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L} \\ &= \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L} \end{aligned} \quad (6)$$

Total energy of the system is

$$E = \int dx T_{00} = \int dx \left[ \frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} (\partial_x \phi)^2 + U(\phi) \right] \quad (7)$$

The vacuum solution is the stationary point of the potential  $U(\phi)$ . The vacuum solution is infinitely degenerate with  $\phi_0 = 2n\pi$  with  $n \in \mathbb{Z}$ . The vacuum solution of the SG model is trivial solution and is not interesting. But there can be solutions interpolating between 2 different vacuum configurations in the 2 extreme ends of the space. For example if we consider, the space to be spanned from 0 to  $L$  i.e.  $x \in [0, L]$  and consider a current of the form,  $j^\mu = \epsilon^{\mu\nu} \partial_\nu \phi$ , then this current is conserved identically [2].

$$\partial_\mu j^\mu = \epsilon^{\mu\nu} \partial_\mu \partial_\nu \phi = 0 \quad (8)$$

This is not a Noether current for any continuous symmetry. If there is no degenerate vacua then this type of current conservation does not contain any interesting information. But for SG model we can define a conserved charge,

$$Q = \frac{1}{4\pi} \int_0^L dx \partial_x \phi = \frac{1}{4\pi} [\phi(L) - \phi(0)] \quad (9)$$

The trivial vacuum solution has  $Q = 0$ , but if the field is interpolating between two different vacua then  $Q \neq 0$ . In general field configurations corresponding to  $Q = +1$  is soliton and  $Q = -1$  is called anti-soliton.  $Q = 0$  sector also has a non-trivial solution called breather. The conserved charge  $Q$  is called topological charge [2].

## SOLITONS AND BREATHERS

Classical SG theory on infinite flat space has exact solutions in terms of quasi-particles which propagates as lump of energy. These solutions correspond to different topological charges. For example,  $Q = 1$  solution is soliton,  $Q = -1$  is called anti-soliton and  $Q = 0$  has non-trivial solution called breather. If we can somehow obtain one solution say  $\phi$  which satisfies the equation of motion(10) then we can generate other solutions from that through Backlund transformation(11). This can be understood in a transformed coordinate  $\tau = \frac{1}{2}(x+t)$  and

$\rho = \frac{1}{2}(x-t)$ . In this transformed coordinate the equation of motion (5) becomes,

$$\partial_\rho \partial_\tau \phi = \sin \phi \quad (10)$$

If we have some field  $\tilde{\phi}$  satisfying the below equation(11) then  $\phi_1 = \phi + \tilde{\phi}$  would be another solution of the equation of motion. We start by trivial solution  $\phi = 0$ . The parameter  $\epsilon$  in equation(11) is related to the velocity of the quasi-particle. These solutions are often called kinks, because the energy is mostly localized in a very narrow region and doesn't spread over time.

$$\begin{aligned} \partial_\tau \phi_1 &= \partial_\tau \phi - 2\epsilon \sin \left( \frac{\phi + \phi_1}{2} \right) \\ \partial_\rho \phi_1 &= \partial_\rho \phi + \frac{2}{\epsilon} \sin \left( \frac{\phi - \phi_1}{2} \right) \end{aligned} \quad (11)$$

The one-kink solution obtained this way is given in equation(12). This has topological charge  $Q = 1$ . The energy( $E$ ) and momentum( $P$ ) of the kink solution are also given. For any arbitrary constant  $\delta$  (position of the peak of the kink at time  $t = 0$ ),

$$\begin{aligned} \phi(\tau, \rho) &= 4 \arctan \exp(\epsilon\tau - \frac{\rho}{\epsilon} + \delta) \\ v &= \frac{1 - \epsilon^2}{1 + \epsilon^2} \\ E &= \frac{8}{\sqrt{1 - v^2}} ; P = \frac{8v}{\sqrt{1 - v^2}} \end{aligned} \quad (12)$$

starting from a kink solution we can obtain kink anti-kink solution by another Backlund transformation with  $\epsilon_2 = -\epsilon$ . The total energy and momentum for this solution is the sum of individual energies and momenta of the kink and anti-kink. Both the solutions are taken to be centered at  $x = 0$  at  $t = 0$ , at large times  $t \rightarrow \pm\infty$  the kinks and anti-kinks are well separated.

$$\begin{aligned} \phi_{k\bar{k}} &= 4 \arctan \frac{\sinh(\gamma vt)}{v \cosh(\gamma x)} \\ \gamma &= \frac{1}{\sqrt{1 - v^2}} \end{aligned} \quad (13)$$

Taking the analytic continuation of parameter  $v$  in the kink anti-kink solution(13) as  $v \rightarrow \frac{i\omega}{\sqrt{1 - \omega^2}}$  we obtain the breather solution which is considered as the bound state of the kink and anti-kink. The kink anti-kink solution and the breather solutions are the non-trivial solutions corresponding to topological charge  $Q = 0$ .

$$\begin{aligned} \phi_B &= 4 \arctan \left( \frac{1 - \omega^2}{\omega^2} \frac{\sin(\omega t)}{\cosh(x\sqrt{1 - \omega^2})} \right) \\ E_\omega &= 16\sqrt{1 - \omega^2} \end{aligned} \quad (14)$$

Solutions of SG theory on finite compact space is different from these, but as we take the length of the space larger and larger, the energy spectrum (for  $Q = 0$ ) asymptotically reaches the breather solutions.

### SINE-GORDON MODEL AS DEFORMED CFT

In QFT the SG model can be treated as an integrable deformation of a free boson CFT [3], [4]. We put the field theory on top of a cylinder to avoid IR divergences. Let the circumference of the cylinder be  $L$ . The action for a free boson on a cylinder is given by,

$$S = \int_{-\infty}^{+\infty} dt \int_0^L dx \left[ \frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} (\partial_x \phi)^2 \right] \quad (15)$$

The stress tensor is given as equation(6). We make a change of coordinates as  $\xi = \frac{1}{\sqrt{2}}(t + ix)$  and  $\bar{\xi} = \frac{1}{\sqrt{2}}(t - ix)$ . The mapping from a cylinder to a complex plane is obtained as weyl transformation of  $\{\xi, \bar{\xi}\}$ . Which is given as  $z = e^{\frac{2\pi\xi}{L}}$ ,  $\bar{z} = e^{\frac{2\pi\bar{\xi}}{L}}$ . This conformal mapping maps infinite past ( $t = -\infty$ ) to origin ( $z = 0$ ) and infinite future ( $t = +\infty$ ) to the point at infinity ( $z = \infty$ ). Conservation laws in  $\{t, x\}$  coordinate is given by,

$$\partial_\mu T^{\mu\nu} = 0 \quad (16)$$

There will be 2 such equations. In  $\{z, \bar{z}\}$  coordinates the stress tensor and the corresponding conservation equations have the form,

$$\begin{aligned} T_{zz} &= \frac{1}{2} \partial_z \phi \partial_z \phi \\ T_{z\bar{z}} &= T_{\bar{z}z} = 0 \\ T_{\bar{z}\bar{z}} &= \frac{1}{2} \partial_{\bar{z}} \phi \partial_{\bar{z}} \phi \\ \partial_z T_{zz} &= 0, \quad \partial_{\bar{z}} T_{\bar{z}\bar{z}} = 0 \end{aligned} \quad (17)$$

Above equation(17) indicate that  $T_{zz}$  is purely holomorphic and  $T_{\bar{z}\bar{z}}$  is anti-holomorphic function in complex plane. Thus  $T_{zz}$  has Laurent expansion. The series solutions are given as,

$$\begin{aligned} T_{zz} = T(z) &= \sum_{k=-\infty}^{\infty} L_k z^{-k} \\ T_{\bar{z}\bar{z}} = \bar{T}(\bar{z}) &= \sum_{k=-\infty}^{\infty} \bar{L}_k \bar{z}^{-k} \end{aligned} \quad (18)$$

The most general solution of the field  $\phi$  in terms of the complex coordinates is given as,

$$\begin{aligned} \phi(z, \bar{z}) &= \Phi(z) + \bar{\Phi}(\bar{z}) \\ \Phi(z) &= \frac{1}{2} \phi_0 + p_+ \ln z + i \sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} \frac{1}{k} a_k z^{-k} \\ \bar{\Phi}(\bar{z}) &= \frac{1}{2} \phi_0 + p_- \ln \bar{z} + i \sum_{\substack{k=-\infty \\ k \neq 0}}^{+\infty} \frac{1}{k} \bar{a}_k \bar{z}^{-k} \end{aligned} \quad (19)$$

We study the field configuration having quasi-periodic boundary condition. The topological charge  $Q$  and the total momentum  $\Pi_0$  are related to two quantum numbers  $n$  and  $m$ , which specify the field configurations. The constants  $\phi_0, p_+, p_-$  are related to these quantum numbers [3].

$$\begin{aligned} Q &= \frac{1}{4\pi} \int_0^L dx \partial_x \phi = m \\ \Pi_0 &= \int_0^L dx \partial_t \phi = 2\pi n \\ p_{\pm} &= n \pm \frac{m}{2} \end{aligned} \quad (20)$$

### PRIMARY OPERATORS AND DESCENDANTS

Primary operator in a CFT is defined by the transformation property of the operator under a general conformal transformation of coordinate system. In  $(1+1)$ -d a local conformal transformation is given by,  $z \rightarrow z' = f(z)$ ,  $\bar{z} \rightarrow \bar{z}' = \bar{f}(\bar{z})$ , where  $f(z)$  and  $\bar{f}(\bar{z})$  are holomorphic and anti-holomorphic functions of  $z$  and  $\bar{z}$  respectively. The transformation property of primary field is given by,

$$X(z', \bar{z}') = \left( \frac{\partial f}{\partial z} \right)^{-h} \left( \frac{\partial \bar{f}}{\partial \bar{z}} \right)^{-\bar{h}} X(z, \bar{z}) \quad (21)$$

The numbers  $h$  and  $\bar{h}$  are called weights of the primary field  $X$ . This values can be determined by considering infinitesimal coordinate transformation and looking at the OPE (operator product expansion) with the stress tensor(17). The two point function in free boson CFT is exactly known. It can be easily checked that the field  $\phi$  is not primary. But  $\partial\phi$  is primary with weights  $(h, \bar{h}) \simeq (1, 0)$ . From the state-operator correspondence in CFT any primary operator  $X$  acting on vacuum is an eigenstate of the Hamiltonian, and has the property,

$$\begin{aligned} X(z, \bar{z}) |0\rangle &= |h, \bar{h}\rangle \\ L_0 |h, \bar{h}\rangle &= h |h, \bar{h}\rangle, \quad \bar{L}_0 |h, \bar{h}\rangle = \bar{h} |h, \bar{h}\rangle \\ L_n |h, \bar{h}\rangle &= \bar{L}_n |h, \bar{h}\rangle, \quad h = 0 \quad \forall n > 0 \end{aligned} \quad (22)$$

Here  $L_n$  are the Virasoro generators on a plane. They form the Virasoro algebra.  $L_n$  for  $n < 0$  acting on  $|h, \bar{h}\rangle$  forms a tower of eigenstates of the Hamiltonian. These are called descendants. The primary along with its descendants together form an overcomplete basis set (overcomplete because there might be null states in some levels) called Verma module. The complete Hilbert space of the whole system is collection of all such modules for all primary operators in the system.

Another non-trivial primary operator in the free boson CFT is the vertex operator. They are given by,

$$\begin{aligned} V_\beta &= \exp[i\beta\phi] \\ V_{(n,m)} &= \exp[i(p_+\Phi(z) + p_-\bar{\Phi}(\bar{z}))] \end{aligned} \quad (23)$$

In the region where topological charge is zero,  $p_+ = p_- = \beta = n$  equation(20). The weights of this primary are  $h = \bar{h} = \frac{\beta^2}{2}$ . Ideally for non-compact space  $\beta$  can take any value from continuous real line. But for free boson on a cylinder due to the boundary condition  $\beta$  can take only integer values.

## TRUNCATED CONFORMAL SPACE APPROACH

The Hamiltonian for the free boson CFT can be written as,

$$H_{CFT} = \frac{2\pi}{L}(L_0 + \bar{L}_0 - \frac{c}{12}) \quad (24)$$

And the Hamiltonian of the SG theory is given in equation(7). Apart from an additional constant 1, this is a perturbation of  $\cos \phi$  potential. However for our case we consider a more general form of the potential  $\frac{1}{\beta^2} \cos \beta\phi$ . The weights of the deformation is  $(\frac{\beta^2}{2}, \frac{\beta^2}{2})$ . The deformation can be written as a sum of two primary operators. The total Hamiltonian [ 3] becomes,

$$\begin{aligned} H &= H_{CFT} + V \\ V &= - \int_0^L dx \cos \phi = - \frac{1}{2} \int_0^L dx [V_{(\beta,0)} + V_{(-\beta,0)}] \end{aligned} \quad (25)$$

Now we take finite number of low energy states from the Hilbert space spanned by the descendants of all primary operators, and calculate the matrix elements of  $H$  (24) in that basis. Diagonalising that matrix would yield the energy spectrum of the theory in low energy sector. The matrix elements of the deformation  $V$  are three point functions in CFT. For free boson the three point function with vertex operators is given as,

$$\begin{aligned} &\langle V_{\beta_1}(z_1)V_{\beta_2}(z_2)V_{\beta_3}(z_3) \rangle \\ &= \delta(\beta_1 + \beta_2 + \beta_3) \prod_{i < j} |z_{ij}|^{\beta_i\beta_j} \end{aligned} \quad (26)$$

Once a three point function of primary operators are known, then the three point functions of the descendants of those primary operators are also known.

$$\begin{aligned} &\langle \{L_{-k}X_1(z_1)X_2(z_2)X_3(z_3)\} \rangle \\ &= \mathcal{L}_{-k} \langle X_1(z_1)X_2(z_2)X_3(z_3) \rangle \\ \mathcal{L}_{-k} &= \sum_{i \neq 1} \left\{ \frac{(k-1)h_i}{(z_i - z_1)^k} - \frac{1}{(z_i - z_1)^{k-1}} \partial_{z_i} \right\} \end{aligned} \quad (27)$$

The deformation operator  $V$  should have a conformal dimension  $\Delta = h + \bar{h} < 2$ . Otherwise the deformation would not be relevant. Couplings with dimension more than 2 will not be renormalizable in (1+1)-d field theory. In our case  $\beta = \frac{1}{2}$ . So the conformal dimension of deformation is  $\Delta = \frac{1}{2}$ . The low energy spectrum as a function of  $r$  (circumference of the cylinder) is obtained as the below figure.

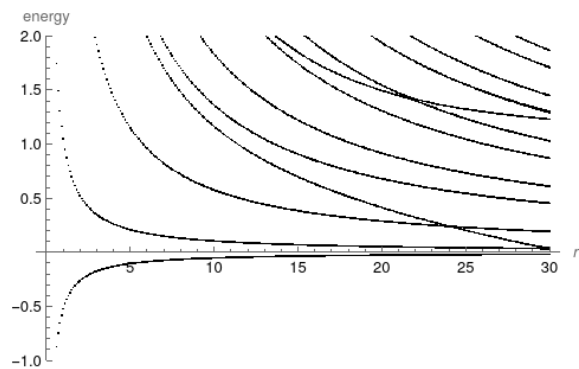


FIG. 1. The TCSA spectrum of the quantum SG model on a cylinder plotted as a function of the system size  $r$  (the circumference of the cylinder).

## CONCLUSION

TCSA is a numerical procedure. We obtain the spectrum numerically. However, SG model on a cylinder can be solved exactly. It is very similar to a 6-vertex model on a closed 2-d euclidean space. We can obtain the exact energy spectrum using thermodynamic Bethe Ansatz(TBA) and corresponding Non Linear Integral Equations(NLIE) [2], [3], [5]. The energy eigenvalues can be given in terms of the roots of the Bethe equations. We can compare those results. Though the SG theory is exactly solvable, certain variations of that (such as double sine gordon model) can not be solved analytically. For 1+1-d field theories TCSA is a reliable method to numerically estimate few low energy eigenstates with great accuracy. TCSA can be used to obtain solutions of many other models. This way of calculating eigenstates numerically might help us to have a deeper

insight into the effect of deformation in any integrable theory.

The mathematica code which is used for above calculations is given [here](#). Part of the code is originally written by [Matthew Headrick](#).

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