Modular Linear Differential Equations to classify Rational Conformal Field Theories

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We will study a special class of 2D Conformal Field Theories(CFTs), which have rational central charge, referred to as Rational Conformal Field Theories(RCFTs). These RCFTs have large applications to statistical models. Subsequently, using modular transformations and modular invariance techniques, we would construct Modular Linear Differential Equations(MLDEs) in order to classify RCFTs on the basis of number of characters of the theory and the number of poles in the coefficients of the MLDE. We would also study *quasi* characters in order to generate RCFTs with a larger number of poles in the coefficients of MLDEs.

I. INTRODUCTION

Rational Conformal Field Theories(RCFTs) are a special class of 2D Conformal field theories(CFTs) that have a finite number of primaries, which results in rational values of central charge [1]. These are used to describe various condensed matter models, such as Fractional Quantum Hall effects [2]. MLDEs are also constructed whose solutions are Virasoro Conformal Blocks which is present in the 4-point function on the Riemann sphere [3]. In order to get a better understanding of different RCFTs, we would study the Modular Linear Differential Equations(MLDEs) of the modular parameter τ whose solutions are the characters of the theory. In this study, we would classify the RCFTs by demanding non-negative integrality of the coefficients in the *q-expansion* of the characters of the theory in order that they may be interpreted as the number of states at each level for the operator associated with the character. The basis of classification would be the number of characters of the RCFT and the number of poles in the coefficients of the MLDEs. In general, MLDEs can have solutions which are called *quasi* characters, which have integral coefficients, however, they need not be non-negative [4]. By adding two *quasi* characters with a small number of poles in the coefficients of the MLDE, we would construct an admissible character that has a larger number of poles in the coefficients of the MLDE.

II. MODULAR FORMS

A function is a modular invariant if

- it is bounded in the upper half complex plane $\mathcal{H}$ of the modular parameter τ, i.e., it is non-singular as $\tau \to i\infty$, and
- it is invariant under modular transformation, i.e.

$$\tau \to \tfrac{a\tau+b}{c\tau+d} \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)$$

Here f is a modular function, which is a modular form with weight zero.

A Modular form of weight k is a function that maps f: $\mathcal{H} \to \mathbb{C}$, and satisfies the equation,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

In the Fundamental Domain(FD), or the moduli space of τ, we have two bases of modular transformation, S and T, where S: $\tau \to -1/\tau$ and T: $\tau \to (\tau+1)$. So a modular form, for these transformations must obey,

$$\mathrm{T} : f(\tau+1) = f(\tau) \qquad \mathrm{S} : f(-\frac{1}{\tau}) = \tau^k f(\tau) \tag{1}$$

A constant function is a modular function. It turns out that there are no non-trivial modular forms with weight zero unless the second condition of boundedness is relaxed as $\tau \to i\infty$. A non-trivial holomorphic, modular function(weight 0) is the Klein j-invariant function j(τ) (details in appendix VII A). Similarly, the Dedekind η function is a modular form of weight 1/2. Eisenstein series E_{2k} are modular forms (k$\in \mathbb{N}\backslash\{1\}$) given by,

$$E_{2k}(\tau) = \frac{1}{2\zeta(2k)} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^{2k}} \tag{2}$$

where $\zeta(2k)$ is the Riemann ζ function.

III. MODULAR LINEAR DIFFERENTIAL EQUATIONS

A MLDE is a linear differential equation that is invariant under modular transformation II. We can construct it by using modular forms of different weights. A simple derivative $\partial/\partial\tau$ acting on a modular form is not covariant

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under modular transformation. So, we define a modular covariant derivative of weight k as,

$$\mathcal{D}_\tau^k = \frac{\partial}{\partial \tau} - \frac{i\pi k}{6} E_2(\tau) \tag{3}$$

where $E_2(\tau)$ is the second Eisenstein series 2 which is not a Modular form but acts as a connection in the moduli space. In analogy with General Relativity, it is analog to the Christoffel symbol, which itself is not a Tensor that transforms in a covariant or contravariant fashion under infinitesimal transformation but acts as a connection term in the covariant derivative. Also, a single Covariant derivative increases the weight of the modular form by 2 (Check Appendix VII B for proof). So the most general modular differential equation can be written as,

$$(f_{2l}\mathcal{D}^n + f_{2l+2}\mathcal{D}^{n-1} + \cdots + f_{2l+2n-2}\mathcal{D} + f_{2l+2n})\chi = 0$$

where f_k are modular forms of weight (-k) and χ is the character of the CFT, and

$$\mathcal{D}^n = \mathcal{D}_{(k+2n-2)} \otimes \mathcal{D}_{(k+2n-4)} \otimes \cdots \otimes \mathcal{D}_{(k+2)} \otimes \mathcal{D}_{(k)} \tag{4}$$

is the covariant derivative, taking a k-weighted modular form to 2n weighted modular form.

The character χ itself is not a modular form, rather it is a *Vector Valued Modular Form*(VVMF),

$$\chi_i(\gamma_\tau) = \sum_k V_{ik}(\gamma)\chi_k(\tau) \tag{5}$$

where V_{ik} are unitary matrices so that under modular transformations II, the character transforms as a linear combination of different characters. This ensures that the Partition function $\mathbb{Z}$ is modular invariant [5]. If the characters were not independent, the V_{ik} wouldn't be unitary. The partition function for a multiplicity M can be written as $\mathbb{Z} = |\chi_0|^2 + M|\chi_1|^2$. If we replace $\chi_1 \to \sqrt{M}\chi_1$, the coefficients of χ_1 will no longer be non-negative integral. This has to be treated separately by considering (M+1)$\times$(M+1) S modular matrix instead of a 2×2 matrix [6](Check Appendix VII D).

A. Upper bound on conformal dimension

We will consider p independent characters, and find the upper bound for the conformal dimension. Consider the master Wronskian,

$$W_p = det \begin{pmatrix} \chi_0 & \chi_1 & \cdots & \chi_{p-1} & \chi \\ \mathcal{D}\chi_0 & \mathcal{D}\chi_1 & \cdots & \mathcal{D}\chi_{p-1} & \mathcal{D}\chi \\ \vdots & \vdots & & \vdots & \vdots \\ \mathcal{D}^{p-1}\chi_0 & \mathcal{D}^{p-1}\chi_1 & \cdots & \mathcal{D}^{p-1}\chi_{p-1} & \mathcal{D}^{p-1}\chi \\ \mathcal{D}^{p}\chi_0 & \mathcal{D}^{p}\chi_1 & \cdots & \mathcal{D}^{p}\chi_{p-1} & \mathcal{D}^{p}\chi \end{pmatrix}$$

where χ is some linear combination of $\chi_0, \chi_1, \ldots, \chi_{p-1}$ all of which are VVMFs 5, which makes $W_p = 0$. We can expand the expression for W_p about the last column about χ,

$$W = \sum_{k=0}^{p} (-1)^{p-k} W_k \mathcal{D}^k \chi = 0 \tag{6}$$

where

$$W_k = det \begin{pmatrix} \chi_0 & \chi_1 & \cdots & \chi_{p-1} \\ \mathcal{D}\chi_0 & \mathcal{D}\chi_1 & \cdots & \mathcal{D}\chi_{p-1} \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^{k-1}\chi_0 & \mathcal{D}^{k-1}\chi_1 & \cdots & \mathcal{D}^{k-1}\chi_{p-1} \\ \mathcal{D}^{k+1}\chi_0 & \mathcal{D}^{k+1}\chi_1 & \cdots & \mathcal{D}^{k+1}\chi_{p-1} \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^{p-1}\chi_0 & \mathcal{D}^{p-1}\chi_1 & \cdots & \mathcal{D}^{p-1}\chi_{p-1} \\ \mathcal{D}^{p}\chi_0 & \mathcal{D}^{p}\chi_1 & \cdots & \mathcal{D}^{p}\chi_{p-1} \end{pmatrix}.$$

W_k is formed out of holomorphic characters and covariant derivatives, so it is a holomorphic modular form of weight (p(p+1) - 2k), as the k^{th} row is removed. $W \equiv W_p$ is a modular form of weight p(p+1) as there are modular covariant derivatives of the order $\frac{p(p+1)}{2}$. The weight $p(p+1) = (2 + 4 + 6 + \cdots + 2(p-1) + 2p)$ is clear from the definition of Modular covariant derivative 4, acting on a VVMF which has weight 0.

We can write the differential equation 6 in a monic form as,

$$\left(\mathcal{D}^p + \sum_{k=0}^{p-1} \phi_k \mathcal{D}^k\right)\chi = 0 \tag{7}$$

where $\phi_k = (-1)^{p-k}\frac{W_k}{W_p}$ are meromorphic modular forms of weight 2(p-k), such as ratio of Eisenstein series(E_6/E_4, E_4^2/E_6, etc.).

As $\tau \to i\infty$, W behaves as $\exp(2\pi i\tau(\sum_i h_i - \frac{pc}{24}))$. It has a pole of the order $(\frac{pc}{24} - \sum_i h_i)$. So the order of zero of W from Riemann Roch Theorem is [7], (check Appendix VII C for the counting of zeros in the Fundamental Domain(FD))

$$-\sum_i h_i + \frac{pc}{24} + \frac{p(p-1)}{12} \tag{8}$$

There can be in general a fractional number of zeros, corresponding to the zeros lying on the boundary of the FD. When there is a zero on the vertical lines of FD, it corresponds to 1/2-order zero, and similarly, a zero at some intersection of two lines with angle θ in between them is $\frac{\theta}{2\pi}$-order zero. It means any zeros of W lying on the vertical line of z = -1/2 + i(Im(z)), it will have a zero at z at z' = 1/2 + i(Im(z)), by T transformation. So it would correspond to 1/2 + 1/2 = 1 order of a zero, as the zeros on these vertical lines account for half of a zero. Similarly, one can have zeros at the cusps. By T transformation, a zero at ρ_1 would imply that there is a zero at ρ_2, where $\rho_1 = e^{2\pi i/3}$ and $\rho_2 = e^{\pi i/3}$ are the cusps or orbifold points [8]. These would correspond to

1/6 of a zero each, just by calculating the angle of the cusp. So a zero at ρ_1 and ρ_2 would correspond to 1/3 of a zero. Also, there can be a zero sitting at z = i, which would correspond to 1/2 of a zero. Any number of zeros lying inside the moduli space would, therefore, sum to an integer number of zeros. Therefore, the total number of zeros of W,

$$-\sum_i h_i + \frac{pc}{24} + \frac{p(p-1)}{12} = \frac{l}{6} \tag{9}$$

where l is an integer ≥ 0. We can then immediately find the upper bound on $\sum_i h_i$ from the above equation 9,

$$\sum_i h_i \leq \frac{p(p-1)}{12} + \frac{pc}{24} \tag{10}$$

A lower bound can be 0 if we are in search of unitary RCFTs, however, in general, the lower bound for a single primary field was found to be $-\frac{2}{5}$ using Fusion rules [8]. We will keep this as a future study work.

IV. MLDE FOR 2 CHARACTER THEORY

A. p=2 and $l = 0$

The most general MLDE with 2 characters and no poles(l=0) in the monic form using eqn. 7,

$$\left(\mathcal{D}^2 + \mu E_4\right)\chi = 0 \tag{11}$$

There is no $\mathcal{D}$ term as we cannot construct a weight 2 modular form with zero poles. Similarly, for the term without the modular derivative, we use E_4 modular form with weight 4 which does not have any poles in the FD. Objects such as the modular covariant derivative $\mathcal{D}$ and Eisenstein series E_4 are modular covariant. The character χ is a VVMF 5. So the equation 11 is modular covariant. Expanding the modular covariant derivative using 3 and 4 we get,

$$\frac{\partial^2 f}{\partial \tau^2} - \frac{i\pi}{3}E_2\frac{\partial f}{\partial \tau} + \mu\pi^2 E_4 f = 0 \tag{12}$$

where E_4 is the Eisenstein series with weight 4 which is a modular form, thus keeping the weight of the differential equation intact, and μ is a constant. The π^2 in front of μ is inserted for convenience, which changes as per the definition of Modular covariant derivative 3 which can include a factor of $2\pi i$ in the denominator if one defines $\partial_\tau \equiv \frac{\partial_\tau}{2\pi i}$. This equation was later renamed to MMS equation, dedicated to Samir D. Mathur, Sunil Mukhi, and Ashoke Sen, for their first attempt to solve it and search for consistent RCFTs [8].

Let us consider a trial solution f which has a *q-expansion*,

$$f = q^\alpha \sum_{n=0}^{\infty} f_n q^n \tag{13}$$

The *q-expansion* coefficients of Eisenstein series 2 are given by,

$$E_k(\tau) = \sum_{n=0}^{\infty} E_{k,n} q^n \tag{14}$$

The coefficients in the *q-expansion* for the character that is associated with the Identity operator($\mathbb{I}$) get restricted as they correspond to the number of secondary states and would be non-negative integers. The vacuum of Quantum Field Theory(QFT) is non-degenerate, and the leading term coefficient is 1 (unless the model is a Weiss-Zumino-Witten (WZW) model, in which case the dimension of the representation of the group that the primary belongs to is the number of states at the lowest level [8]), and the rest would be non-negative integers. However, for the character associated with the non-identity operator (ϕ), this need not be the case, as the vacuum can be degenerate, however, all other coefficients have to be non-negative integers, as they represent the number of secondary states associated with the primary. Substituting the above *q-expansions* of trial solution 13 and Eisenstein series 14 in the second order MLDE 12 gives,

$$4\alpha^2 - \frac{2}{3}E_{2,0}\alpha - \mu E_{4,0} = 0 \tag{15}$$

$$f_n = -(4n\alpha + 2n^2 - \frac{1}{3}E_{2,0}n)^{-1}\sum_{k=1}^{n}\{-\frac{1}{3}E_{2,k}(n+\alpha-k) - \frac{\mu}{2}E_{4,k}\}f_{n-k} \tag{16}$$

Also, considering normalized Eisenstein series 14, we have $E_{k,0} = 1 \forall k$, we get α from the quadratic equation for α 15,

$$\alpha = \frac{1}{12}(1 \pm \sqrt{1+36\mu}) \equiv \frac{1}{12}(1 \pm x) \tag{17}$$

where x is the positive square root of 1+36μ. The smaller value of $\alpha = \frac{1}{12}(1-x)$ is associated with the Identity operator. So $\frac{1-x}{12} = -\frac{c}{24}$. So the central charge is c = 2(x-1). Inserting value of α from eqn. 17 in eqn. 16, to get a recursion relation, we extract the ratio of f_1 and f_0,

$$m_1 \equiv \frac{f_1}{f_0} = \frac{10x^2 + 2x - 12}{6 - x}$$

Inverting the above equation we get,

$$x = \frac{-(m_1+2) \pm \sqrt{(m_1+2)^2 + 40(12+6m_1)}}{20}$$

The restriction that we have a finite number of primaries in the theory results in the fact that the central charge c and conformal dimensions of the primaries h_i are rational numbers [1], whose proof we are keeping for future study.

Now as the central charge is rational, we expect $(m_1 + 2)^2 + 40(12 + 6m_1) = k^2$, where k is an integer. Define

$$s = 120 + (m_1 + 2) - k$$

We can write the relation between m_1 and k as,

$$m_1 + 2 = \frac{(120 - s)^2}{2l} = \frac{(120)^2}{2s} - 120 + \frac{s}{2}$$

LHS being an integer, we expect s to be even and is a divisor of $\frac{120^2}{2}$.

Similarly, if we use the other character which is associated with the non-identity operator, we get $\frac{(1+x)}{12} = (h - \frac{c}{24})$. Using $c = 2(x - 1)$, we get the conformal dimension h = $\frac{x}{6}$.

For example, consider $m_1 = 1$, which gives x = $\frac{6}{5} > 1$, thus c = 2(x-1) = $\frac{2}{5}$. The exponents $\alpha = \frac{1 \pm x}{12} = \frac{11}{60}$ and $-\frac{1}{60}$, respectively. However, there is no such minimal model with a central charge $\frac{2}{5}$ [9]. We obtained this solution by equating $\frac{-1}{60}$ with $\alpha_0 = -\frac{c}{24}$ and $\frac{11}{60}$ with $\alpha_1 = h - \frac{c}{24}$. If we equate $\frac{11}{60}$ with $\frac{-c}{24}$ rather, or interchange c ↔ c - 24h, we get c = $\frac{-22}{5}$ with $h = \frac{-1}{5}$, which is the famous non unitary Lee-Yang Theory [8]. This shows that the RCFTs realized from the admissible cases can be non-unitary as well because our assumption was only characters are independent of each other.

One can also check whether the admissible theory is realized as a consistent RCFT by calculating the fusion rules which are necessarily non-negative integers [10]. It turns out that they are negative for the first case(c = $\frac{2}{5}$), while non-negative for the second case(c = $\frac{-22}{5}$), although it is a non-unitary theory.

The interchanging of the α solution from 17, we did to get an admissible solution, we used a non-unitary representation ($\alpha_1 = h - \frac{c}{24} < \alpha_0 = -\frac{c}{24}$). Even if we get a consistent theory from the unitary representation which is ($\alpha_0 < \alpha_1$), it might result in a non-unitary theory, as the unitary presentation just uses guarantees that $h \geq 0$, but c can be negative. However, if we get a consistent theory out of the non-unitary presentation, it has to be a non-unitary theory as $h \leq 0$ [4].

There are 9 more values of m_1 for which the characters are associated with RCFTs. Most of them correspond to WZW models [8].

One could have integrated the MLDE, instead of using *q-expansion* and searched for an admissible solution. For 1 character theories even have an explicit form of the solution in terms of Klein-j invariant [8] given by,

$$\chi(\tau) = j^{w_\rho}(j - 1728)^{w_i} P_{w_\tau}(j)$$

where j is the Klein-j invariant (Check Appendix VII A), $P_{w_\tau}(j)$ is a polynomial of j of degree w_τ with integral coefficients, $w_\rho = 0, 1/3$ or $2/3$ and $w_i = 0$ or $1/2$. We can get the central charge by using Riemann Roch theorem (p=1) 9,

$$c = 24(w_\rho + w_i + w_\tau)$$

But the analog to two-character theory does not exist [4]. So, it is preferable to start with a power series ansatz. There is an exact expression for the characters with p=2 and $l = 0$, calculated by changing variables to Picard λ function, whose solutions turn out to be Hypergeometric functions [11] (Check Appendix VII D).

Solving these Differential equations can give admissible solutions demanding non-negative integral coefficients in the *q-expansion*, however, this is not sufficient criteria. We have to check the non-negative condition for the Fusion Rules coefficients [10](Check Appendix VII D) and find the chiral algebra [12] as well.

B. p=2 and $l > 0$

In this section, we will construct MLDE for 2 characters but with non-zero fractional zeros of the Wronskian, equivalently, poles in the coefficients of the MLDE.

For $l = 1$, we cannot have 1/6-order zero because a zero at a cusp at $\rho_1 = e^{2\pi i}/3$, would correspond to a zero at $\rho_2 = e^{\pi i}/3$(by T transformation), making it a 1/3-order zero. So there are no 2 character theories with $l = 1$.

For $l = 2$, we have 1/3 of a zero, which is a zero at $\rho_1 = e^{2\pi i}/3$ and $\rho_2 = e^{\pi i/3}$. The Eisenstein series E_4 has zeros at these cusps, thus, we can put a single power of E_4 in the denominator at max. The modular form of weight 4 has to be E_4 as there are no other modular forms with a denominator with E_4 in it. So, the MLDE for 2 characters with $l = 2$,

$$\left(\mathcal{D}^2 + \mu_1 \frac{E_6}{E_4}\mathcal{D} + \mu_2 E_4\right)\chi = 0 \tag{18}$$

Similarly for $l = 3$, we put a single power of E_6, as it has a zero at z = i. So the MLDE for l=3,

$$\left(\mathcal{D}^2 + \mu_1 \frac{E_4^2}{E_6}\mathcal{D} + \mu_2 E_4\right)\chi = 0 \tag{19}$$

For l=4, we need 2/3 of a zero, which is $4 \times \frac{1}{6}$ of a zero, for which we need E_4^2 in the denominator. We can now construct a term with weight 4, i.e., $\frac{E_6^2}{E_4^2}$. So the MLDE for l=4 is,

$$\left(\mathcal{D}^2 + \mu_1 \frac{E_6}{E_4}\mathcal{D} + \mu_2 E_4 + \mu_3 \frac{E_6^2}{E_4^2}\right)\chi = 0 \tag{20}$$

For l=5, in the denominator we can have terms from E_6 (1/2-order zero), E_4(1/3-order zero) and E_4E_6 (5/6-order zero). However, we cannot even construct terms of weight 2 or 4 that would give 5/6-order zero as E_4 and E_6 are the two basis of the modular forms we are using. So the MLDE for l=5 is,

$$\left(\mathcal{D}^2 + \left(\mu_1 \frac{E_6}{E_4} + \mu_2 \frac{E_4^2}{E_6}\right)\mathcal{D} + \mu_3 E_4\right)\chi = 0 \tag{21}$$

For $l \geq 6$, we can use a zero lying completely inside the moduli space, but with the cost of increasing the number of parameters μ_i. For this, one generally uses the method of *quasi* characters [4] (studied in next section V), whose coefficients are integers, but can be negative as well. The addition of two quasi-characters can lead to an admissible character with a higher l value than that of the individual quasi-characters.

V. QUASI CHARACTERS

Quasi characters are *quasi* in the sense that their coefficients are integers, but need not be non-negative. There are two types of quasi-characters, namely Type I and Type II. Type I quasi-characters are the characters that have a finite number of negative integer coefficients and the rest all are non-negative. Type II quasi-characters have a finite number of non-negative integer coefficients the rest being negative integers. While each of these quasi-character characters does not correspond to admissible characters, however, one can add a Type I and a Type II quasi-character in such a way as to make all the coefficients non-negative. If we have an identity character,

$$\chi = q^{-\frac{c}{24}}(a_0 + a_1 q + a_2 q^2 + \cdots) \tag{22}$$

with $a_0 > 0$. Type I characters have finite negative coefficients in the form, $a_0, a_1, a_2, \cdots, a_n$ are ≤ 0, while the rest $a_{n+1}, a_{n+2}, \cdots$ are > 0. Similarly, Type II characters have $a_0, a_1, a_2, \cdots, a_m \geq 0$ and while the rest $a_{m+1}, a_{m+2}, \cdots$ are < 0. We shall consider a type I and a type II character of this type and construct an admissible solution by suitable addition of the characters.

A. Kaneko-Zagier parametrisation

Kaneko and Zagier studied a variant for the MMS equation 12 in [13], which was for any general modular form with weight k,

$$\left(\mathcal{D}^2_{(k)} - \frac{k(k+2)}{144}E_4(\tau)\right) f_{(k)}(\tau) = 0 \tag{23}$$

Substituting $f_{(k)}(\tau) = \eta(\tau)^{2k}(\tau)\chi(\tau)$, we get back the MMS equation

$$\left(\mathcal{D}^2 - \frac{k(k+2)}{144}E_4(\tau)\right)\chi(\tau) = 0$$

with $\mu = -\frac{k(k+2)}{144}$. Using 17, we get c = 2k. Out of all the admissible solutions of the MMS equation 12, we have $-2 \leq c \leq 10$, which implies $-1 \leq k \leq 5$. For $k = \frac{1}{5}$, we get c = $\frac{2}{5}$, we get the Lee Yang series. Any values outside this range will give quasi characters [4].

The Lee Yang series a Particular cases like $k = \frac{6n+1}{5}$, where $n \neq 4 \bmod 5$, correspond to the Lee Yang series. We get c = 2k = $\frac{2(6n+1)}{5}$ and $h = \frac{n+1}{5}$ for $l = 0$ and $p = 2$. To add characters, they also need to have the same Modular S matrix. For n=0 and n=10, it turns out that they both have the same Modular S matrix which is given in 33. They correspond to quasi-characters with $l - 0$.

We compute $\chi_i^{n=10} + N_1\chi_i^{n=0}$ using the $l = 0$ quasi characters from [4]. Indices i=0,1 correspond to the character associated with the identity and the non-identity operator, respectively. For i=0,

$$\begin{aligned} &q^{-\frac{61}{60}}(1 - 244q + 169641q^2 + \cdots) + N_1 q^{-\frac{1}{60}}(1 + q + q^2 + ...) \\ &\quad = q^{-\frac{61}{60}}(1 + (N_1 - 244)q + (N_1 + 169641)q^2 + \cdots) \end{aligned} \tag{24}$$

and for i=1,

$$\begin{aligned} &q^{\frac{71}{60}}(310124 + 27523505q + \cdots) + N_1 q^{\frac{11}{60}}(1 + q + q^2 + ...) \\ &\quad = q^{\frac{11}{60}}(N_1 + 310124q + (N_1 + 27523505)q^2 + \cdots) \end{aligned} \tag{25}$$

We will get an admissible solution for $N_1 \geq 244$. We can extract the central charge and conformal dimension h by equating $\alpha_0 = \frac{-61}{60} = -\frac{c}{24}$ and $\alpha_1 = h - \frac{c}{24}$, which gives c = $\frac{122}{5}$ and h = $\frac{6}{5}$. From the Riemann Roch theorem 9, we get l = 6. So we were able to generate an admissible solution with $l = 6$ starting from two $l = 0$ quasi characters, each for $N_1 \geq 244$.

VI. DISCUSSIONS AND CONCLUSION

In this study, we studied the most general Modular Invariant Linear Differential Equation(MLDE), whose solutions are the characters of the theory. We were able to classify Rational CFTs(RCFTs) in search of admissible characters by demanding non-negative integrality coefficients in the *q-expansion* of the characters. We also constructed an admissible solution with $l = 6$ starting from two $l = 0$ *quasi* characters.

Future Outlook: The mapping of modular functions and elliptic curves is of great interest to Number theorists. We might look into some concepts from Number theory, such as Elliptic curves that can be used to interpret results in Modular forms in RCFTs. Some results of Number theory are in Appendix VII A.

We can also search if other Hall Effects can also emerge out of classifying RCFTs [14]. We can at least perform numerical simulations with large systems, approaching the thermodynamic limit to match results with the free fields in RCFTs.

We would like to study the MLDEs for one point function or N-point function using the Kaneko-Zagier MLDE 23 [15],[16].

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VII. APPENDIX

A. Number theory

A. Klein-j invariant function is given by,

$$j(\tau) = \frac{\left(1 + 240\sum_{n\geq 0}\sigma_3(n)q^n\right)^3}{q\prod_{n>0}(1-q^n)^{24}}$$

where $\sigma_3(n) = \sum_{d|n} d^3$ is the divisor series of power 3. It has an *q-expansion* as,

$$j(\tau) = \frac{1}{q} + 744 + 196844q + 21493760q^2 + \cdots \tag{26}$$

The number 196883 turns out to be the dimension in which the Monster group lives.

Also from the number theory result of the Ramanujan-Sato series of $\frac{1}{\pi}$, we have a factor of $(640320)^{3/2}$ sitting in the denominator. Also, it is a known result that $e^{\pi\sqrt{163}}$ turns out to be an *almost* integer, close to $262537412640768743.99999\cdots$. Now in Modular forms, using the first two terms from the above *q-expansion* 26 of the Klein-j invariant about $\left(\frac{1+\sqrt{-163}}{2}\right)$, it turns out remarkably that,

$$j\left(\frac{1+\sqrt{-163}}{2}\right) = -262537412640768000 = -(640320)^3$$

where we used ($q = -e^{-\pi\sqrt{163}} \sim 0$). We get this esoteric relation of number theory result from the Ramanujan-Sato series with that of Klein-j Invariant modular form.

B. Consider the Discriminant $\Delta(q) = \eta(q)^{24}$, which has *q-expansion*,

$$\Delta(q) = q\prod_n(1-q^n)^{24} = q - 24q^2 + 252q^3 - 1452q^4 + 4830q^5 - 6048q^6 + \cdots$$

Writing it in Ramanujan's notation, $\Delta(q) = \sum q^n\tau(n)$, it turns out, $\tau(mn) = \tau(m)\tau(n)$, if (m,n) = 1, i.e. they are coprime. These are obtained using Hecke operators, which are fundamental operations using Modular Forms [17].

B. Modular Covariant Derivative

If f is a modular form of weight k, we need to show that the quantity $\mathcal{D}^k_\tau f(\tau)$ is a modular form of weight (k+2). Then using the definition of Modular covariant derivative 3,

$$\mathcal{D}^k_{\gamma_\tau} f(\gamma_\tau) = \left((c\tau+d)^2\partial_\tau - \frac{i\pi k}{6}E_2(\gamma_\tau)\right)(c\tau+d)^k f(\tau) \tag{27}$$

We will use a result for the Modular transformation of the E_2 series given by,

$$G_2(\gamma_\tau) = (c\tau+d)^2 G_2(\tau) - \frac{c(c\tau+d)}{4\pi i}$$

where $G_2(\tau) = -\frac{E_2(\tau)}{24}$ [18].

For T transformation, $\tau \to \tau+1$, we have a = 1, b = 1, c = 0 and d = 1. Plugging these values in the above equation of Modular covariant derivative 27,

$$\begin{aligned}\mathcal{D}^k_{\gamma_\tau} f(\gamma_\tau) &= \left(\partial_\tau - \frac{i\pi k}{6}E_2(\tau+1)\right) f(\tau)\\ &= \left(\partial_\tau - \frac{i\pi k}{6}E_2(\tau)\right) f(\tau)\\ &= \mathcal{D}^k_\tau f(\tau)\end{aligned} \tag{28}$$

For S transformation, $\tau \to -\frac{1}{\tau}$, we have a = 0, b = -1, c = 1 and d = 0. Plugging these values in the above equation of Modular covariant derivative 27,

$$\begin{aligned}\mathcal{D}^k_{\gamma_\tau} f(\gamma_\tau) &= \left(\tau^2\partial_\tau - \frac{i\pi k}{6}E_2(\frac{-1}{\tau})\right)\tau^k f(\tau)\\ &= \tau^{2+k}\partial_\tau f(\tau) + k\tau^{1+k} f(\tau)\\ &- \frac{i\pi k}{6}(\tau^2 E_2(\tau) + \frac{6\tau}{\pi i}) f(\tau)\tau^k\\ &= \tau^{2+k}\partial_\tau f(\tau) + k\tau^{k+1} f(\tau)\\ &- \frac{i\pi k}{6}\tau^{2+k}E_2(\tau) f(\tau) - k\tau^{k+1} f(\tau)\\ &= \tau^{2+k}\left(\partial_\tau - \frac{i\pi k}{6}E_2(\tau)\right) f(\tau)\\ &= \tau^{2+k}\left(\mathcal{D}^k_\tau f(\tau)\right)\end{aligned} \tag{29}$$

These modular transformations follow the original definition of modular transformation 1.

C. Riemann Roch

If we have poles in the Fundamental domain, we can compute the contour integral along the boundaries to evaluate the residue. We know from Complex analysis, that the contour integral of $\frac{df}{f}$ gives the (number of poles - number of zeros) of the function. We would use this integral to evaluate the number of zeros of f which is a modular form of weight k, in the FD by demanding that there are no singularities of the characters in the FD.

The contribution from the vertical lines $1/2 + iy$ and $-1/2 + iy$ but in opposite direction gives zero, as $f(\tau) = f(\tau+1)$ from T transformation.

The poles at $\tau \to i\infty$ is computed by taking the contour $q = 1/2 + iy$ to $q = -1/2 + iy$ as $y \to \infty$. This contour in q plane is a contour which is a circle around the origin (q=0). Integrating the function $\frac{df}{f}$ gives,

$$\frac{1}{2\pi i}\int \frac{df}{f} = -(\text{Number of zeros at } \tau \to i\infty)$$

For the circular part of the contour(in the clockwise direction and hence, an extra minus sign), lying between the two cusps $\rho_1 = e^{2\pi i/3}$ and $\rho_2 = e^{\pi i/3}$, the S Transformation maps the contour $\rho_1 \to i$ to $i \to \rho_2$, almost canceling the integral but leaving a term coming from the τ^k from the S transformation 1,

$$\int \frac{df}{f} = \frac{1}{2\pi i}\int_{\frac{1}{12}\text{ circle}} \frac{d\tau^k}{\tau^k} = \frac{k}{12}$$

Thus, the number of zeros of a modular form f with weight k in FD is $\frac{k}{12}$, which includes the zeros at $\tau \to i\infty$.

D. Modular matrix and Fusion rules

There is an exact solution for the characters for the case p=2 and l=0 in terms of hypergeometric functions [11] using the transformation $\lambda(\tau) : \tau \to \frac{\vartheta_2^4(\tau)}{\vartheta_3^4(\tau)}$, where λ is the Picard-λ function and $\vartheta_{2,3}$ are Jacobi theta function.

The S and T transformation 1 in the Picard λ function corresponds to $\lambda \to 1-\lambda$ and $\lambda \to \frac{\lambda}{\lambda-1}$ respectively.

In the basis where the T transformation modular matrix is diagonal,

$$T = \begin{pmatrix} e^{2\pi i(h-c/24)} & 0\\ 0 & e^{2\pi i(-c/24)}\end{pmatrix} \tag{30}$$

This is consistent with the T transformation 1, as it attaches only a phase to the characters so that the power series ansatz 13 is an eigenstate of the T modular matrix.

From S transformation in λ, we can compute the S matrix, as $S\begin{pmatrix}\chi_1(\lambda)\\ \chi_0(\lambda)\end{pmatrix} = \begin{pmatrix}\chi_1(1-\lambda)\\ \chi_0(1-\lambda)\end{pmatrix}$,

$$S = \begin{pmatrix} \frac{\Gamma(1-\frac{1}{3}x)\Gamma(\frac{1}{3}x)}{\Gamma(\frac{1}{2}-\frac{1}{6}x)\Gamma(\frac{1}{2}+\frac{1}{6}x)} & \frac{(16)^{\frac{x}{3}}}{N}\frac{\Gamma(1-\frac{1}{3}x)\Gamma(-\frac{1}{3}x)}{\Gamma(\frac{1}{2}-\frac{1}{6}x)\Gamma(\frac{1}{2}-\frac{1}{2}x)}\\ \frac{N}{(16)^{\frac{x}{3}}}\frac{\Gamma(1+\frac{1}{3}x)\Gamma(\frac{1}{3}x)}{\Gamma(\frac{1}{2}+\frac{1}{6}x)\Gamma(\frac{1}{2}+\frac{1}{2}x)} & \frac{\Gamma(1+\frac{1}{3}x)\Gamma(-\frac{1}{3}x)}{\Gamma(\frac{1}{2}+\frac{1}{6}x)\Gamma(\frac{1}{2}-\frac{1}{6}x)}\end{pmatrix} \tag{31}$$

where x is defined in 17. From the above equation 31, we can use the Verlende formula [10] to compute the Fusion Rules coefficients $\mathcal{N}_{ijk}$.

$$\mathcal{N}_{ijk} = \sum_{m=0}^{1}\frac{S_{im}S_{jm}S_{km}}{S_{0m}} \tag{32}$$

where S is the modular S matrix 31. For c = $\frac{2}{5}$ [6],

$$S = \begin{pmatrix}\sqrt{\frac{2}{5-\sqrt{5}}} & \sqrt{\frac{2}{5+\sqrt{5}}}\\ \sqrt{\frac{2}{5+\sqrt{5}}} & \sqrt{\frac{2}{5-\sqrt{5}}}\end{pmatrix} \tag{33}$$

which gives $N_{000} = N_{011} = 1$, but $N_{111} = -1$.

However, considering c= $-\frac{22}{5}$ and h = $-\frac{1}{5}$, we get $N_{000} = N_{011} = N_{111} = 1$.