Momentum space Ward identities and connection with dS

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This report provides a comprehensive exploration of the concept of momentum space Ward identities. The study begins by establishing an understanding of it and elucidating the intricate interplay between symmetries and correlation functions in physical systems. Central to this discussion is the role of momentum space Ward identities in inflationary models, particularly in the context of conformal field theories. An elucidation of their contribution to the understanding of the early universe, providing insights into the dynamics of scalar fields and their impact on cosmic evolution is provided in this report.

INTRODUCTION

The Momentum Space Ward Identity presents a comprehensive investigation into the fundamental properties and implications of Ward identities in the context of quantum field theory. The formulation and analysis of Ward identities in momentum space, provide a novel perspective that offers unique insights into the behaviour of physical systems.
Conformal invariance imposes strong constraints on correlation functions. It determines two- and three-point functions of scalars, conserved vectors and the stress-energy tensor. For example,
$n = 2$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{c_{12}}{|x_1 - x_2|^{2\Delta}} \tag{1}$$

$n = 3$

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_n(x_3)\rangle = \frac{c_{123}}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3 - x_1|^{\Delta_3+\Delta_1-\Delta_2}} \tag{2}$$

$n > 3$

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = f(u,v)\prod_{1\le i<j\le n} x_{ij}^{\delta_{ij}} \tag{3}$$

where $x_{ij} = |x_i - x_j|$ are coordinate separation

$$2\delta_{ij} = \frac{\Delta_t}{3} - \Delta_i - \Delta_j, \qquad \Delta_t = \Sigma_{i=1}^n \Delta_i \tag{4}$$

u and v and conformal cross ratios are defined as

$$u = \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{23}^2}, \qquad v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \tag{5}$$

The findings are straightforward to obtain when using position space, where the conformal group behaves naturally. However, for various contemporary uses like cosmology, condensed matter physics, anomalies, and the bootstrap program, it's crucial to understand the equivalent of this result in momentum space.

WARD IDENTITIES

The infinitesimal transformation ϵ_μ is at most quadratic in x_μ given by $\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho$
The generator of the conformal transformations are
(Translation) $P_\mu = -i\partial\mu$
(Rotation) $M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$
(Dilatation) $D_\mu = -ix^\mu\partial_\mu$
(Special Conformal) $K_\mu = -i(2x_\mu x_\nu\partial_\nu - x^2\partial_\mu)$

Transformation of correlation functions by assuming conformal invariance of action is given by [1]

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} ... \left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\Delta_n/d} \langle \mathcal{O}_1(x_1')...\mathcal{O}_n(x_n')\rangle \tag{6}$$

For infinitesimal dilatations($\delta x^\mu = \lambda x^\mu$) this yields

$$0 = \left[\sum_{j=1}^n \Delta + \sum_{j=1}^n x_j^\alpha \frac{\partial}{\partial x_j^\alpha}\right]\langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle \tag{7}$$

In momentum space this becomes

$$0 = \left[\sum_{j=1}^n \Delta_j + (n-1)d - \sum_{j=1}^n p_j^\alpha \frac{\partial}{\partial p_j^\alpha}\right]\langle \mathcal{O}_1(p_1)...\mathcal{O}_n(p_n)\rangle \tag{8}$$

For infinitesimal Special conformal transformation ($\delta x^\mu = b^\mu x^2 - 2x^\mu(b\cdot x)$) this yields

$$0 = \left[\sum_{j=1}^n \left(2\Delta_j x_j^\kappa + 2x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} - x_j^2 \frac{\partial}{\partial x_{jk}}\right)\right]\langle \mathcal{O}_1(p_1)...\mathcal{O}_n(p_n)\rangle \tag{9}$$

In momentum space this becomes

$$0 = \mathcal{K}_\mu\langle \mathcal{O}_1(p_1)...\mathcal{O}_n(p_n)\rangle$$
$$\mathcal{K}_\mu = \left[\sum_{j=1}^{n-1}\left(2(\Delta_j - d)\frac{\partial}{\partial p_j^\kappa} + 2p_j^\kappa \frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_j^\kappa} - (p_j)_\kappa \frac{\partial}{\partial p_j^\alpha}\frac{\partial}{\partial p_{\alpha j}}\right)\right] \tag{10}$$

where $\mathcal{K}_\mu$ is a second-order differential operator independent of the tensor structure. There will be a tensorial

part in the case of tensor operators. Special Conformal Ward identities constitute $(n-1)$ non-linear partial differential equation.
All momenta are not independent. The momentum-conserving delta function relates them. for example

$$\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\rangle = (2\pi)^3\delta^3(p_1+p_2)\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\rangle\prime$$

Solution of Ward Identities

The solution for the two-point correlation function is

$$\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\rangle\prime = \int d^d\boldsymbol{x}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\frac{1}{x^{2\Delta}} = \frac{\pi^{d/2}2^{d-2\Delta}\Gamma(\frac{d-2\Delta}{2})}{\Gamma(\Delta)}p^{2\Delta-d} \tag{11}$$

The integral converges for $0 < 2\Delta < d$.

The solution for the three-point correlation function can be represented in triple-K integral form[2]:

$$\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(p_3)\rangle\prime = C_{123}p_1^{\Delta_1-\frac{d}{2}}p_2^{\Delta_2-\frac{d}{2}}p_3^{\Delta_3-\frac{d}{2}} \int dx x^{\frac{d}{2}-1}K_{\Delta_1-\frac{d}{2}}(p_1x)K_{\Delta_2-\frac{d}{2}}(p_2x)K_{\Delta_3-\frac{d}{2}}(p_3x) \tag{12}$$

where $K_\nu(p)$ is a modified Bessel function of second kind and C_{123} is a constant which appear in operator product expansion(OPE).
The solution for the higher-point function is given in terms of *simplex*[3] which is computationally hard and not discussed here.

CONNECTION WITH DESITTER

Conformal invariance plays a crucial role in the study of inflationary cosmology. Inflation proposes that the universe underwent a phase of exponential expansion, driven by a scalar field known as the inflaton. It explains the homogeneity and isotropy of the early Universe at large scales.

Conformal symmetry puts constraints on cosmological perturbations during inflation. The Universe during inflation was approximately de-Sitter, with a symmetry group $SO(4,1)$ which is same as that of a *3 dimensional Euclidean CFT*. This is somewhat like dS/CFT correspondence.

The de-Sitter spacetime has a metric of the form:

$$ds^2 = -dt^2 + e^{2Ht}\sum_{i=1}^{3}dx_i^2 \tag{13}$$

where $-\infty < t, x_i < \infty$ and the constant H is the de-Sitter scale or the Hubble parameter.

The conformal time η is given by $\eta = -\frac{1}{H}e^{-Ht}$, then metric becomes

$$ds^2 = -\frac{1}{H^2\eta^2}(d\eta^2 + \sum_{i=1}^{3}dx_i^2)$$

The isometry group of $dS_4, SO(4,1)$, has 10 Killing generators: 3 translations, 3 rotations, a scale transformation of the form [4] $x^i \to \lambda x^i$, $t \to t - \frac{1}{H}log(\lambda)$
and 3 special conformal transformations whose infinitesimal form is
$x^i \to x^i - 2(b_jx^j)x^i + b^i\big(x_jx^j - \frac{e^{-2Ht}}{H^2}\big)$,
$t \to t + 2\frac{b_jx^j}{H}$

The Euclidean $AdS4(EAdS4)$ also has the same symmetry group $SO(3,2)$ as $dS4$ and upon analytic continuation they can be transformed to each other.
The simplest model of inflation, known as the slow-roll model, is based on the dynamics of a slowly varying scalar field coupled to two-derivative gravity. The model has three key parameters namely Hubble Constant H, and two slow-roll parameters ϵ and δ that play a crucial role in determining the evolution of the universe during the inflationary phase.
The $SO(4,1)$ symmetry of dS_4 is slightly broken due to the slow rolling of the inflaton. [5] The action for single field slow-roll models of inflation is given by

$$S = \frac{M_{pl}^2}{2}\int dx_4\sqrt{-g}\big(R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2V(\phi)\big) \tag{14}$$

In the homogeneous limit, the inflaton is purely a function of time,$\phi \equiv \bar{\phi}(t)$ and the metric is the unperturbed FRW metric.

The slow-roll conditions are imposed by setting the slow-roll parameters ϵ_1, δ_1 to be much less than unity, where

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \delta_1 = \frac{\ddot{H}}{2H\dot{H}} \tag{15}$$

The slow-roll criterion $\epsilon_1, \delta_1 \ll 1$ ensures that the universe is not exactly deSitter during the inflationary phase.
The slow-roll conditions can also be expressed in terms of the slow-roll parameters ϵ, δ where

$$\epsilon = -\frac{1}{2}\frac{\dot{\bar{\phi}}^2}{H^2}, \delta = \frac{\ddot{\bar{\phi}}}{H\dot{\bar{\phi}}} \tag{16}$$

For canonical slow-roll model $\epsilon_1, \delta_1 = \epsilon, \delta$ respectively. The Potential slow-roll parameters are defined by

$$\epsilon_v = \frac{1}{2}\bigg(\frac{V'}{V}\bigg)^2, \delta_v = \bigg(\frac{V''}{V}\bigg) \tag{17}$$

In the slow roll approximation, they are related by $\epsilon_v = \epsilon_1$ and $\eta_v = \epsilon_v = \epsilon_1 - \delta$

The ADM formalism which is used at an instant time is well suited for the Hamiltonian construction of general relativity, the metric in terms of the shift and lapse functions, N_i and N respectively, as

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{18}$$

The metric of dS space is rotationally invariant with SO(3) symmetry in the x^i directions. This invariance can be used to classify the perturbations. There are two types of perturbations, scalar($\delta\phi$, ζ) and tensor(γ_{ij}), which transform as spin 0 and spin 2 under the rotation group respectively.

$$\begin{aligned} h_{ij} &= e^{2Ht}(\delta_{ij}(1+2\zeta) + \gamma_{ij}), \gamma_{ii} = 0 \\ \phi &= \bar{\phi} + \delta\phi \end{aligned} \tag{19}$$

where ϕ, h_{ij} are the dynamical variables, whereas N, N_i are Lagrange multipliers.

A convenient choice is the *synchronous gauge*, $N = 1$, $N_i = 0$ and using the remaining gauge redundancy of time and spatial parametrization at late enough times to consider transverse-traceless gauge for the graviton, i.e. $\partial_i\gamma_{ij} = 0$and the scalar perturbations can also be chosen such that either $\delta\phi = 0$ and $\zeta \neq 0$ or $\phi \neq 0$ and $\zeta = 0$. It is important to note that we will work in momentum space and through position space analysis of the correlation functions are known in a CFT, it is not straightforward to convert them to momentum space due to the presence of *contact terms.*

Symmetry considerations are useful when studied in terms of wave function of the Universe, correlation functions can be calculated from the wave function as a functional of the scalar and tensor perturbations. It is defined as a functional of the late time values of the perturbations χ ($\delta\phi$, γ_{ij}) through the path integral [6]

$$\Psi[\delta\phi, \gamma_{ij}] = \int^{\delta\phi,\gamma_{ij}} D\chi e^{S[\chi]} \tag{20}$$

We will be interested in the wave function at late times when the modes of interest have crossed the horizon so that their wavelength $\lambda \gg$ H, this boundary condition is particularly called *Bunch-Davies vacuum.* So, they were insensitive to the curvature of the de-Sitter space and essentially propagated as if in Minkowski space. Expanding at late times, when the perturbations become time-independent, we get

$$\begin{aligned} \Psi[\gamma_{ij}] = exp\Big[&-\frac{1}{2}\int d^3xd^3y\zeta(x)\zeta(y)\langle T(x)T(y)\rangle \\ &-\int d^3xd^3y\zeta(x)\hat{\gamma}_{ij}(y)\langle T(x)\hat{T}^{ij}(y)\rangle \\ &-\frac{1}{2}\int d^3xd^3y\hat{\gamma}_{ij}(x)\hat{\gamma}_{kl}(y)\langle \hat{T}^{ij}(x)\hat{T}^{kl}(y)\rangle \\ &-\frac{1}{3!}\int d^3xd^3yd^3z\zeta(x)\zeta(y)\zeta(z)\langle T(x)T(y)T(z)\rangle \\ &+ ... \\ &-\frac{1}{m!n!}\int d^3x_1...d^3x_{m+n}\zeta(x_1)...\zeta(x_m) \\ &\hat{\gamma}_{i_1j_1}(x_{m+1})...\hat{\gamma}_{i_1j_1}(x_{m+n}) \\ &\times \langle T(x_1)...T(x_m)\hat{T}^{i_1j_1}(x_{m+1})...\hat{T}^{x_{m+n}}(y)\rangle + ...\Big] \end{aligned} \tag{21}$$

where

$T = 2T_{ii}$ related to trace of the stress-energy tensor, so that the coefficient function for a general metric perturbation, γ_{ij}, is T^{ij}.
$\hat{T}_{ij}$ is the traceless part of the stress-energy tensor T_{ij}.

The quadratic terms in ζ and γ_{ij} correspond to a Gaussian wave function; higher order terms give rise to non-Gaussianity.

The expectation values for the perturbations are obtained from the wave function in the standard manner. For example, for scalar perturbations ζ these are given by

$$\langle\zeta(x_1)...\zeta(x_n)\rangle = \int \frac{1}{N}[\mathcal{D}\zeta][\mathcal{D}\gamma_{ij}]|\Psi|^2\zeta(x_1)...\zeta(x_n) \tag{22}$$

Ward identities for Scale Transformation

[7] Under a scale Transformation

$$x^i \to x^i + \lambda x^i, \lambda \ll 1 \tag{23}$$

Correspondingly ζ and $\hat{\gamma}_{ij}$ transforms as

$$\begin{aligned} \zeta &\to \zeta + \lambda + \lambda x^i\partial_i\zeta \\ \hat{\gamma}_{ij} &\to \hat{\gamma}_{ij} + \lambda x^k\partial_k\hat{\gamma}_{ij} \end{aligned} \tag{24}$$

It is worth noticing that the traceless condition is preserved while transformation. On changing variables, the measure is spatial reparameterization invariant. The wave function Ψ is invariant under this transformation, leading to the condition

$$\langle\delta(\zeta(x_1))...\zeta(x_n)\rangle + ...\langle\zeta(x_1)...\delta(\zeta(x_n))\rangle = 0 \tag{25}$$

which gives an incorrect transformation of ζ i.e $\zeta \to \zeta + \lambda x^i\partial_i\zeta$

The homogenous term in transformation gives rise to quadratic in ζ which cancels from the piece of Ψ cubic in ζ .
After carefully handling the quadratic term which will give a linear piece in ζ from the inhomogeneous term in the transformation ($\delta\zeta = \lambda + ..$) is

$$\delta\Psi \sim exp\Big(- \lambda \int d^3x d^3y \zeta(x)) \langle T(x)T(y)\rangle \Big) \tag{26}$$

Not eliminating the linear term gives us the right Ward identity

$$\begin{aligned} &\langle \delta(\zeta(x_1))...\zeta(x_n)\rangle + ...\langle \zeta(x_1)...\delta(\zeta(x_n))\rangle = \\ &2\lambda \int d^3x d^3y \zeta(x))\langle T(x)T(y)\rangle\langle \zeta(x_1)...\zeta(x_n)\zeta(x)\rangle \end{aligned} \tag{27}$$

Dropping the linear piece in λ on the l.h.s., leading to

$$\begin{aligned} &\Big[\sum_{a=1}^{n} x_a \cdot \frac{\partial}{\partial x_a}\Big]\langle \zeta(x_1)...\zeta(x_n)\rangle = \\ &2 \int d^3x d^3y \zeta(x))\langle T(x)T(y)\rangle\langle \zeta(x_1)...\zeta(x_n)\zeta(x)\rangle \end{aligned} \tag{28}$$

In momentum space we have

$$\begin{aligned} &\Big(3(n-1) + \sum_{a=1}^{n} k_a \cdot \frac{\partial}{\partial k_a}\Big)\langle \zeta(k_1)...\zeta(k_n)\rangle\prime = \\ &- \frac{1}{\zeta(k_{n+1})\zeta(-k_{n+1})\prime}\langle \zeta(k_1)...\zeta(k_{n+1})\rangle\prime\Big|_{k_{n+1}\to 0} \end{aligned} \tag{29}$$

where $\langle \zeta(k_1)\zeta(k_2)\rangle = (2\pi)^3\delta^3(k_1+k_2)\langle \zeta(k_1)\zeta(k_2)\rangle\prime$

Similarly, for tensor perturbation $\hat{\gamma}_{ij}$ the correlation function is

$$\begin{aligned} &\Big[\sum_{a=1}^{n} x_a \cdot \frac{\partial}{\partial x_a}\Big]\langle \hat{\gamma}_{i_1 j_1}(x_1)...\hat{\gamma}_{i_1 j_1}(x_n)\rangle = \\ &2 \int d^3x d^3y \langle T(x)T(y)\rangle\langle \hat{\gamma}_{i_1 j_1}(x_1)...\hat{\gamma}_{i_1 j_1}(x_n)\zeta(x)\rangle \end{aligned} \tag{30}$$

The correlation function in momentum space becomes,

$$\begin{aligned} &\Big(3(n-1) + \sum_{a=1}^{n} k_a \cdot \frac{\partial}{\partial k_a}\Big)\langle \hat{\gamma}_{i_1 j_1}(k_1)...\hat{\gamma}_{i_1 j_1}(k_n)\rangle\prime = \\ &- \frac{1}{\zeta(k_{n+1})\zeta(-k_{n+1})\prime}\langle \hat{\gamma}_{i_1 j_1}(k_1)...\hat{\gamma}_{i_1 j_1}(k_n)\zeta(k_{n+1})\rangle\prime\Big|_{k_{n+1}\to 0} \end{aligned} \tag{31}$$

Eq.(29) and Eq.(31) are Maldacena consistency conditions i.e. bulk and boundary calculation would match. The l.h.s. of eq. (28) is the change of the n-point correlator under an overall change of scale. Exactly such a transformation is generated by a scalar perturbation $\zeta(n+1)$ in the limit of very long wavelength, $k_{n+1} \to 0$, leading to the identity eq. (29).

In the perfect de-Sitter space the r.h.s. of the Ward identity will become zero, which is evident from the expressions on page 1.

These cosmological perturbations are related to the anisotropies in the CMB radiation and provided seeds for the structure formation in this Universe. More specifically the scalar perturbations ($\delta\phi, \zeta$) are related to temperature anisotropies and the tensor one (γ_{ij}) are related to the polarization of the CMB radiation.

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[1] A. Bzowski, P. McFadden and K. Skenderis, JHEP **03**, 111 (2014) doi:10.1007/JHEP03(2014)111 [arXiv:1304.7760 [hep-th]].

[2] A. Bzowski, Comput. Phys. Commun. **258**, 107538 (2021) doi:10.1016/j.cpc.2020.107538 [arXiv:2005.10841 [hep-th]].

[3] A. Bzowski, P. McFadden and K. Skenderis, Phys. Rev. Lett. **124**, no.13, 131602 (2020) doi:10.1103/PhysRevLett.124.131602 [arXiv:1910.10162 [hep-th]].

[4] N. Kundu, A. Shukla and S. P. Trivedi, JHEP **04**, 061 (2015) doi:10.1007/JHEP04(2015)061 [arXiv:1410.2606 [hep-th]].

[5] A. Shukla, S. P. Trivedi and V. Vishal, JHEP **12**, 102 (2016) doi:10.1007/JHEP12(2016)102 [arXiv:1607.08636 [hep-th]].

[6] N. Kundu, PhD Thesis https://inspirehep.net/files/6b4e8a789a3bffecf5f8de21dab0e738

[7] N. Kundu, A. Shukla and S. P. Trivedi, JHEP **01**, 046 (2016) doi:10.1007/JHEP01(2016)046 [arXiv:1507.06017 [hep-th]].