Causality Constraints in CFT

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Causality imposes significant limitations on Quantum Field Theory (QFT), one of which is leading to the determination of the signs of specific terms in the low-energy Lagrangian. In the context of d-dimensional Conformal Field Theory, we illustrate how these constraints manifest through the principle of crossing symmetry and reflection positivity in Euclidean correlation functions. Additionally, we can analytically derive equivalent constraints by applying the techniques of the conformal bootstrap.

INTRODUCTION

Quantum field theories that exhibit causal propagation when analyzed in vacuum may exhibit violations of causality in non trivial states. Imposing the condition of causality in all possible states places restrictions on the permissible interactions within the theory. It is evident that the constraints associated with causality are fundamentally linked to the Lorentzian signature.

In the first section, we talk about the holographic motivation behind this project. The next section gives a pedagogical review of causality in position-space quantum field theory. Proceeding that, we review the conformal bootstrap in Euclidean signature and discuss its extension to Lorentzian signature with timelike separated points.

HOLOGRAPHIC MOTIVATION

After the discovery of AdS/CFT correspondence, there has been several studies in which physicists tried to find correspondence to CFT in different areas, specifically using crossing symmetry. For example, the effective field theories in the bulk are in one-to-one correspondence with solutions of crossing symmetry in CFT, order by order in 1/N, or for each flat-space scalar S-matrix, there is a corresponding crossing-symmetric CFT correlator in Mellin space. In the bulk, causality dictates that certain interactions come with a fixed sign. This property can also be translated, by using crossing symmetry, into a constraint on CFT data, giving a holographic motivation for the topic. To do that, we first need to study the constraints that causality can bring on CFT, which is our goal for this project.

CAUSALITY REVIEW

To be causal, all the spacelike operators shall commute, i.e, $[O_1(x)O_2(y)] = 0, \forall (x-y)^2 > 0$

This report is a study of scalar four point function $\langle \psi O O \psi \rangle$

In this section we will review how the commutator requirement is encoded in the analytic structure of correlation functions, first in a general Lorentz-invariant QFT and then in CFT.

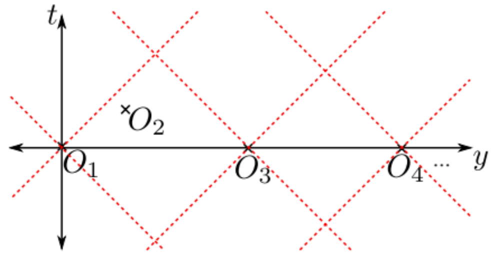

FIG. 1: All t_i's zero except t_2

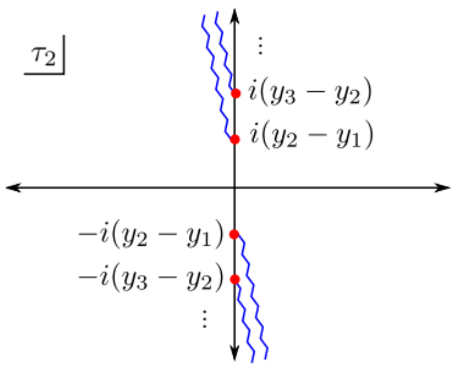

FIG. 2: Singularities (red dots) are branch points, and branch cuts (blue) are oriented almost vertical

Euclidean and Lorentzian Correlators

The Euclidean correlator, denoted by G is
$G(x_1, ..., x_n) = \langle O_1(x_1), ..., O_n(x_n) \rangle$

By performing an analytical continuation $\tau_i \to it_i$ we can compute Lorentzian correlators. When expressed as functions of the complex τ_i, these correlators exhibit a complex network of singularities and branch cuts, introducing ambiguities in the analytic continuation which are responsible for non-vanishing commutators.

In Figure 2, the correlator, when considered as a function of τ_2 while keeping all other parameters fixed, displays singularities aligned with the imaginary axis of τ_2. These singularities occur precisely at the points where the oper-

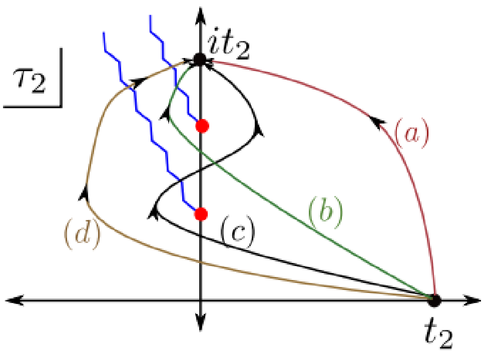

FIG. 3: Possible contours

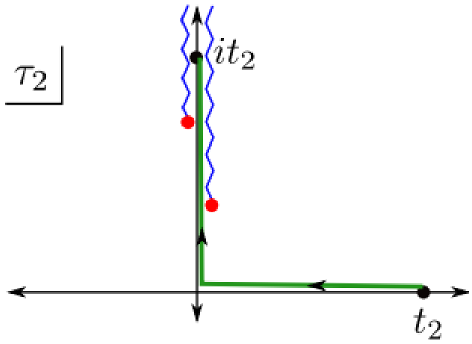

FIG. 4: $i\epsilon$ prescription for (c) contour

ator O_2 intersects the light cones of the other operators. Each time we encounter one of these singularities, we must make a decision on whether to transition to the right or to the left. Assuming $t_2 > 0$, passing to the right makes operators into time ordering and left puts it in anti-time order.

The contours shown in the figure 3 correspond to the following Lorentzian correlators:
(a) $\langle O_2 O_1 O_3 ... \rangle = \langle T[O_1 O_2 O_3 ...] \rangle$
(b) $\langle O_3 O_2 O_1 ... \rangle$
(c) $\langle O_1 O_2 O_3 ... \rangle$
(d) $\langle O_1 O3 O_2 ... \rangle$
As we pass a branch cut, we move to another sheet on which the location of next singularity might be shifted. If it shifts upward, the theory faces a time delay which is fine, but if it shifts downwards, the time is advanced and points which are acausal appear to be causal. To avoid this situation, we need to put constraints in such a way that the next singularity never moves downwards.

$i\epsilon$ prescription

The Osterwalder-Schrader reconstruction theorem states that well behaved Euclidean correlators, upon analytic continuation, result in Lorentzian correlators that obey the Wightman axioms. The results were extended to CFT by Luscher and Mack. A byproduct of these reconstruction theorems is a simple $i\epsilon$ prescription to compute Lorentzian correlators, with any ordering, from the analytically continued Euclidean correlators:

$$\langle O_1(t_1, \vec{x_1}) ... O_n(t_n, \vec{x_n}) \rangle = \lim_{\epsilon_j \to 0} \langle O_1(t_1 - i\epsilon_1, \vec{x_1}) ... O_n(t_n - i\epsilon_n, \vec{x_n}) \rangle$$

where $\epsilon_1 > \epsilon_2 > ... > \epsilon_n > 0$

Example: Conformal 2-point function

The Euclidean 2-point function in CFT is $(\tau^2 + x^2)^{-\Delta}$. Using the $i\epsilon$ prescription, the Lorentzian correlators for $t_1 > x_1$ are
$\langle O(t_1, x_1) O(0,0) \rangle = \exp(-\Delta \log(-(t_1 - i\epsilon)^2 + x_1^2))$
$= e^{-i\pi\Delta}(t_1^2 - x_1^2)^{-\Delta}$
$\langle O(0,0) O(t_1, x_1) \rangle = \exp(-\Delta \log(-(t_1 + i\epsilon)^2 + x_1^2))$
$= e^{i\pi\Delta}(t_1^2 - x_1^2)^{-\Delta}$

CFT 4 PT FUNCTIONS

We now specialize to 4-point functions in a conformal field theory. Take the operators O_1, O_3 and O_4 to be fixed and spacelike separated at $\tau = 0$, while O_2 is inserted at an arbitrary time:
$x_1 = (0, ..., 0)$
$x_2 = (\tau_2, y_2, 0, ..., 0)$
$x_3 = (0, 1, ..., 0)$
$x_4 = (0, \infty, 0, ..., 0)$
with $0 < y_2 < \frac{1}{2}$

The conformal cross ratios are

$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z},$

$v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1 - z)(1 - \bar{z})$

For our case, this becomes $z = y_2 + i\tau_2, \bar{z} = y_2 - i\tau_2$

In Euclidean, τ_2 is real and $\bar{z} = z^*$ and in Lorentzian z, $\bar{z}$ are independent real numbers.

The Euclidean correlator $G(z, z^*)$ has singularities
$G(z, z^*) \sim (zz^*)^{-\frac{1}{2}(\Delta_1 + \Delta_2)}$ as $z \to 0$
$G(z, z^*) \sim ((1 - z)(1 - z^*))^{-\frac{1}{2}(\Delta_2 + \Delta_3)}$ as $z \to 1$

To get the Lorentzian correlators, we do analytic continuation $\tau_2 \to it_2$. Denoting $G(z, \bar{z})$ as the time-ordered correlator for contour (a) of figure 3, we have
(a) $G(z, \bar{z}) = \langle O_2 O_1 O_3 O_4 \rangle$
(b) $G(z, \bar{z})|_{(\bar{z}-1) \to e^{-2\pi i}(\bar{z}-1)} = \langle O_3 O_2 O_1 O_4 \rangle$
(c) $G(z, \bar{z})|_{z \to e^{-2\pi i} z} = \langle O_1 O_2 O_3 O_4 \rangle$
(d) $G(z, \bar{z})|_{z \to e^{-2\pi i} z, (\bar{z} - \bar{z_0}) \to e^{-2\pi i}(\bar{z} - \bar{z_0})} = \langle O_1 O_3 O_2 O_4 \rangle$

These follow from the fact that the first singularity above the real axis is z=0 and the second in $\bar{z} = 1$

Here, in the (d) contour, as we encounter the branch cut of first singularity we move to another sheet on which the second singularity has shifted to $\bar{z}_0$. To ensure the constraint that it should never shift downwards as discussed earlier, it should satisfy-

$$\text{Re } \bar{z}_0 \geq 1$$

THE LORENTZIAN OPE

In this section, we provide an overview of the Euclidean OPE in the context of a d-dimensional CFT. We then deduce certain outcomes stemming from the principle of reflection positivity and explore the extent to which the OPE can be employed in Lorentzian correlation functions. In the scenario where only a single operator exhibits spacelike separation from the rest, we demonstrate the existence of a convergent OPE channel and employ it to establish a connection between causality and reflection positivity.

Conformal block expansion

OPE in CFT is $O_1(x_1)O_2(x_2) = \Sigma f_{12k}(x_1 - x_2)O_k(x_2)$ Applied inside a 4-point correlation function, the OPE gives the conformal block expansion as

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)\rangle = \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{12}} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{34}} \Sigma c_{12p} c_{34p} g^{\Delta_{12},\Delta_{34}}_{\Delta_p,l_p}(z,\bar{z})$$

The Euclidean z-expansion

Consider a 4-point function with two species of operators :
$G(z,\bar{z}) = \langle \Psi(0)O(z,\bar{z})O(1)\Psi(\infty)\rangle$

s channel: OPE $O(z,\bar{z}) \to \Psi(0)$
$G(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_o+\Delta_\psi)} \Sigma_p c_{\psi O p} c_{O\psi p} g^{\Delta_{\Psi O},-\Delta_{\Psi O}}_{\Delta_p,l_p}(z,\bar{z})$
The sum converges for Euclidean points $\bar{z} = z^*$ with $|z| < 1$.

t channel: OPE $O(z,\bar{z}) \to O(1)$
$G(z,\bar{z}) = ((1-z)(1-\bar{z}))^{-\Delta_o} \Sigma_p c_{oop} c_{\psi\psi p} g^{0,0}_{\Delta_p,l_p}(1-z, 1-\bar{z})$
This is obtained by relabeling $x_1 \longleftrightarrow x_3$. This sum converged for Euclidean points with $|1-z| < 1$.

u channel: OPE $O(z,\bar{z}) \to \Psi(\infty)$
$G(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_o-\Delta_\psi)} \Sigma_p c_{O\psi p} c_{\psi O p} g^{\Delta_{\Psi O},-\Delta_{\Psi O}}_{\Delta_p,l_p}(\frac{1}{z}, \frac{1}{\bar{z}})$
This sum is convergent for Euclidean $|z| > 1$

Problem: For a given z, only 2 of the 3 expansions converge and s and u channels have no overlapping range of convergence. This is overcome by the ρ expansion.

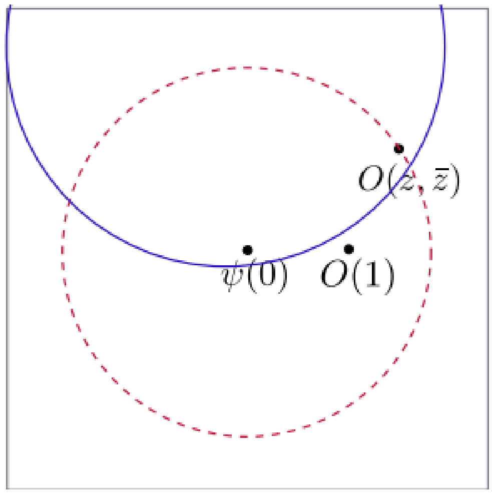

FIG. 5: Idea for implementation of ρ variable

The Euclidean ρ expansion

Figure 5 shows how picking different origin of the z-plane instead of z=0, we can always find a circle that encloses $\psi(0)$ and $O(z,\bar{z})$ without hitting any other operators. Choosing the middle of this circle as the origin for radial quantization will give a convergent expansion. Taking ρ as a complex number with $|\rho| < 1$, we define

$$H(\rho,\bar{\rho}) = \langle \psi(-\rho)O(\rho)O(1)\psi(-1)\rangle$$

Let the summation term in the conformal block expansion formula be S, then we can write
$H(\rho,\bar{\rho}) = \frac{1}{(2\rho)^{\Delta_\psi+\Delta_o}(2)^{\Delta_\psi+\Delta_o}} \left(\frac{\rho+1}{\rho-1}\right)^{\Delta_\psi-\Delta_o} \left(\frac{\rho-1}{\rho+1}\right)^{\Delta_o-\Delta_\psi} S$
$= \left(\frac{\rho+1}{\rho-1}\right)^{2(\Delta_\psi-\Delta_o)} \frac{1}{(4\rho)^{\Delta_\psi+\Delta_o}} S$
$= \left[\frac{(1+\rho)(1+\bar{\rho})}{(1-\rho)(1-\bar{\rho})}\right]^{\Delta_{\psi O}} \frac{1}{(16\rho\bar{\rho})^{\frac{1}{2}(\Delta_O+\Delta_\psi)}} S$

In the s channel, $G(z,\bar{z})$ was calculated previously as
$G(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_O+\Delta_\psi)} S$
$= (z\bar{z})^{-\frac{1}{2}(\Delta_O+\Delta_\psi)} H(\rho,\bar{\rho}) \left[\frac{(1+\rho)(1+\bar{\rho})}{(1-\rho)(1-\bar{\rho})}\right]^{-\Delta_{\psi O}} \frac{1}{(16\rho\bar{\rho})^{-\frac{1}{2}(\Delta_O+\Delta_\psi)}}$
$= \left(\frac{z\bar{z}}{16\rho\bar{\rho}}\right)^{-\frac{1}{2}(\Delta_\psi+\Delta_O)} \left[\frac{(1+\rho)(1+\bar{\rho})}{(1-\rho)(1-\bar{\rho})}\right]^{-\Delta_{\psi O}} H(\rho,\bar{\rho})$

The $\psi(-\rho)O(\rho)$ OPE converges inside the 4-point function for any $|\rho| < 1$, which maps to the full z plane, minus the line $[1,\infty]$. Similarly, on calculating for all three channels, we get the convergence for Euclidean $z \in \mathbf{C}\backslash[1,\infty]$

POSITIVE COEFFICIENTS

$G(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_\psi+\Delta_O)}\Sigma_{h,\bar{h}\geq 0} a_{h,\bar{h}} z^h \bar{z}^{\bar{h}}$
where $h = \frac{1}{2}(\Delta \pm l), \bar{h} = \frac{1}{2}(\Delta \mp l)$

We want to know the sign of the coefficient $(a_{h,\bar{h}})$ to bound the magnitude of the correlator and to know when the expansion converges in Lorentzian (where z, $\bar{z}$ are independent).
Lets define a state $|f\rangle$ in radial quantization.

$$|f\rangle = \int_0^1 dr_1 \int_0^{2\pi} d\theta_1 r_1^{\Delta_\psi+\Delta_O} f(r_1,\theta_1) O(r_1 e^{i\theta_1}, r_1 e^{-i\theta_1})\psi(0)\,|0\rangle$$

In radial quantization, conjugation acts on operators by inversion across the unit sphere as
$[O(z,\bar{z})]^\dagger = (z\bar{z})^{-\Delta_o} O^\dagger\left(\frac{1}{z},\frac{1}{\bar{z}}\right)$

Hence the conjugate becomes
$\langle f| = \langle 0|\,\psi(\infty)\int_0^1 dr_2 \int_0^{2\pi} d\theta_2 r_2^{\Delta_\psi-\Delta_O} f^*(r_2,\theta_2) O\left(\frac{1}{r_2}e^{i\theta_2}, \frac{1}{r_2}e^{-i\theta_2}\right)$

Reflection positivity says $\langle f|f\rangle > 0$, hence
$\int_0^1 dr_1 \int_0^1 dr_2 \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 r_2^{-2\Delta_o}(r_1 r_2)^{(\Delta_\psi+\Delta_O)}$
$f(r_1,\theta_1) f^*(r_2,\theta_2)\,\langle\psi(0) O(x,x^*) O(y,y^*)\psi(\infty)\rangle > 0$
where $x = r_1 e^{i\theta_1}, y = \frac{1}{r_2}e^{i\theta_2}$

By a conformal transformation, the 4-point function in the integrand can be related to the canonical insertion points as
$\langle\psi(0)O(x,x*)O(y,y*)\psi(\infty)\rangle = r_2^{2\Delta_o} G\left(\frac{x}{y},\frac{x*}{y*}\right)$

$$= r_2^{2\Delta_o}\left(\frac{r_1^2}{\frac{1}{r_2^2}}\right)^{-\frac{1}{2}(\Delta_o+\Delta_\psi)} \Sigma_{h,\bar{h}\geq 0} a_{h,\bar{h}} (r_1 r_2 e^{i(\theta_1-\theta_2)})^h (r_1 r_2 e^{-i(\theta_1-\theta_2)})^{\bar{h}}$$

$$= \frac{r_2^{\Delta_o-\Delta_\psi}}{r_1^{\Delta_o+\Delta_\psi}}\Sigma_{h,\bar{h}\geq 0} a_{h,\bar{h}} (r_1 r_2)^{h+\bar{h}} e^{i(\theta_1-\theta_2)(h-\bar{h})}$$

Putting this in the reflection positivity condition -
$\int_\epsilon^{1-\epsilon} dr_1 \int_\epsilon^{1-\epsilon} dr_2 \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 f(r_1,\theta_1) f^*$
$(r_2,\theta_2)\Sigma_{h,\bar{h}\geq 0} a_{h,\bar{h}} (r_1 r_2)^{h+\bar{h}} e^{i(\theta_1-\theta_2)(h-\bar{h})} > 0$

$\Longrightarrow \Sigma_{h,\bar{h}\geq 0} a_{h,\bar{h}} \left|\int_0^1 dr \int_0^{2\pi} d\theta r^{h+\bar{h}} e^{i(h-\bar{h})\theta} f(r,\theta)\right|^2 > 0$
$\Longrightarrow \mathrm{a}_{h,\bar{h}} > 0$ (with a few in between steps skipped)

In a similar way we can show that $b_{h,\bar{h}} > 0$ in
$H(\rho,\bar{\rho}) = (16\rho\bar{\rho})^{-\frac{1}{2}(\Delta_o+\Delta_\psi}\Sigma_{h,\bar{h}\geq 0} b_{h,\bar{h}} \rho^h \bar{\rho}^{\bar{h}}$

CONCLUSION

In this project, on extending the study of causality from QFT to CFT, we found several constraints on the OPE coefficients, conformal block expansion and location of singularities. The first constraint that we studied was that the next singularity should always shift upward in another sheet leading to time delay and not time advancement. The second major result was to find a region where s, t and u channel converge altogether. In our example, this was found to be the whole complex axis except the region greater than 1. The third and strongest constraint found was that the full correlator has an expansion with each coefficient being always positive.

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