2D Turbulence $\simeq$ 2-D Conformal Field Theory

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Turbulence, a universal phenomenon, has perplexed scientists across various scales, from everyday fluids to celestial bodies. The Navier-Stokes equation governs turbulent dynamics, challenging us due to its lack of exact solutions. A. Kolmogorov's pioneering work established statistical insights, such as energy spectra, for 3D and 2D turbulence. This review spotlights A. Polyakov's work [1], which unites 2D Conformal Field Theory (CFT) and 2D turbulence. By considering 2D CFT as isomorphic to 2D turbulence in the inertial range, Polyakov not only reconfirms Kolmogorov's energy spectrum but also unveils some exact results. Exploring this merger between two seemingly distinct fields, we unveil the hidden harmony between turbulent complexities and the elegance of conformal field theory, promising profound insights into the interconnected worlds of fluid dynamics and field theory.

Introduction: Turbulence is a state of the fluid flow where the system is highly chaotic. This is a very fundamental phenomenon, and it can be observed from everyday fluids to celestial bodies. The force law of the fluids goes by the name of the Navier-Stokes Equation (1). We have the inertial forcing (*non-dissipative*), viscous force (*dissipative*) and the Driving force (*stirring*). The viscous coefficient ν, when non-dimensionalised, is the inverse of the Reynolds Number(Re). Fluids with very high Reynolds Number show turbulence. Therefore, $\nu \to 0$ is the limit of turbulence.

$$\partial_t \mathbf{v} + \left(\mathbf{v}\cdot\vec{\nabla}\right)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{F}_{ext}, \tag{1}$$

There is a slight subtlety in the above statement because it is a limit, not the exact value of the ν. It follows from the fact that we have a source term in (1), and if ν is set to zero, then the energy of the system will keep on increasing, and eventually, it will diverge. But if we have a limit, then no matter at what scale the dissipation will be effective, in the end, the energy will be dissipated from the system, and the energy won't diverge. Therefore, we have two scales in the system. The infrared scale is where the Forcing is done, and the UV scale is where the dissipation occurs. We will eventually see that these will regularise our theory.

$$\dot{\omega} + (\mathbf{v}\cdot\nabla)\omega = (\omega\cdot\nabla)\mathbf{v} + \nu\nabla^2\omega, \tag{2}$$

$$\dot{\omega} + e_{\alpha\beta}\partial_\alpha\psi\partial_\beta\partial^2\psi = \nu\partial^2\omega, \tag{3}$$

Our main focus will be 2D turbulent flows, and the form of the Navier-Stokes Equation that we will be dealing with is called the vorticity equation. To obtain this equation, we need to take the curls of (1), and we will obtain (2), where $\omega = \nabla\times\mathbf{v}$. In 2D, using the stream function, we can write it as (3).

$$v_x = -\frac{\partial\psi}{\partial y}, \quad v_y = \frac{\partial\psi}{\partial x}, \quad e_{\alpha\beta} = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \tag{4}$$

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The stream function is defined in 2D flows for incompressible fluids. It is defined as (4). Just by looking at the definition, our previous statement is justified. Finally, in (3), $e_{\alpha\beta}$ is an antisymmetric matrix (4).

Inertial Range: Kolmogorov's turbulence theory posits that small-scale structures exhibit statistical properties of homogeneity, isotropy, and independence from large-scale structures. Under these assumptions, he derived the energy spectrum of the 3D turbulent fluids as $E(k) \propto k^{-5/3}$ and in 2D it is $E(k) \propto k^{-3}$. The range in which this relation holds is called as the inertial region. The term inertial comes because we are away from the length scale where the forcing is done and the viscous scale where the dissipation is done, leaving only the inertial force term on the RHS of (1).
Moreover, the numerical simulations yield a slightly different result. Instead of -3, it gives exponents ranging from -3 to -4. Hence, it can be expected that the equivalent field theory won't be unique.

Hopf equation: It is believed that at the high Reynolds Number, we are dealing with the statistically stationary regime; therefore, the probability distribution is time-independent.

$$\langle\omega(x_1)\omega(x_2)\omega(x_3)...\omega(x_N)\rangle, \tag{5}$$

$$\begin{aligned}&\langle\dot{\omega}(x_1)\omega(x_2)...\omega(x_N)\rangle + \langle\omega(x_1)\dot{\omega}(x_2)...\omega(x_N)\rangle\\ &+ ... \langle\omega(x_1)\omega(x_2)...\dot{\omega}(x_N)\rangle = 0\end{aligned} \tag{6}$$

As a consequence of this, correlations like (5) are time-independent. This results in the obvious result, The Hopf equation [2]. It looks like (6). It is an important result because we can use (2) to substitute $\dot{\omega}$ in (6), and this will relate the N point functions with the N+1 point functions.
Using CFT, we will try to satisfy (6) exactly, assuming that the developed turbulence in the inertial range possesses conformal symmetry. No formal proof says that the fluids will have a conformal symmetry. This is somewhat a kind of ansatz that we put. The argument that makes our assumption reasonable is that the

enstrophy cascade in the inertial range is independent of the length scale, and, hence, the turbulent dynamics must be the same at each scale.

To compile the fluids discussion, we can say that to trust the CFT, which we will call the isomorphic theory, must satisfy (6), and it should not be violated near the regulating scales. Also, we will also see that the Energy Spectrum formula $E(k) \propto k^{-3}$ is satisfied, but not necessarily exactly the same; the exponent could have a value ranging from -3 to -4. Remember, this is the best we can do to test our field theory, and it was mentioned by Polyakov in [1] that whatever the model we will be discussing in the later sections is not "The theory" for the turbulence in 2D. There can be multiple field theories that satisfy these conditions and are the possible contenders. It is even possible that the model will fail for some constraints that we do not know as of now. Keeping this in mind, let's delve into the CFT.

Minimal Model: The minimal models are the class of CFTs with a finite number of primary operators. These primaries have their own *Verma Modules*.

$$c = 1 - 6\frac{(p-q)^2}{pq}, \quad h_{r,s}(p,q) = \frac{(pr-qs)^2-(p-q)^2}{4pq}, \tag{7}$$

The (p,q) minimal model in CFT has the central charge and highest weight of the *Verma Modules* for the primary $\phi_{r,s}$ are (7).Here, $1 \leq r \leq q-1$ and $1 \leq s \leq p-1$. Now, there is a further classification of the minimal models, whether they are unitary or non-unitary. In the unitary theory, the norm of each state is positive semi-definite, or it is non-unitary. If in (7) $p = m+2$, $q = m+3$ with $m \geq 1$, then the theory is unitary, or else it is non-unitary.
The OPE for these models are highly constrained because of the unitarity condition. Following is the fusion rule for the minimal models.

$$\left[\phi_{(p_1,q_1)}\right] \times \left[\phi_{(p_2,q_2)}\right] = \sum_{\substack{k=1+|p_1-p_2| \\ k+p_1+p_2 \ odd}}^{p_1+p_2-1} \ \sum_{\substack{l=1+|q_1-q_2| \\ l+q_1+q_2 \ odd}}^{q_1+q_2-1} \left[\phi_{(k,l)}\right], \tag{8}$$

First Contender: We have all the tools to deal with the problem of turbulence in the inertial limit using 2D CFT. We will consider the (2,5) minimal model. The central charge for such a theory is $c = -22/5$. It is a non-unitary theory with the two primaries with the complex dimensions (0,0) and (-1/5,-1/5). Now we map $\psi \longleftrightarrow$ *primary(-1/5,-1/5).*

$$\langle\psi(\mathbf{x})\psi(0)\rangle = -|\mathbf{x}|^{4/5}, \tag{9}$$

Before going on further, let us look at the two-point functions. In position space, it is (9). To obtain the momentum space representation of the two-point function, we must first regularise the theory to avoid the contract terms.

$$\langle\psi(\textbf{-k})\psi(\mathbf{k})\rangle = \frac{const}{|\mathbf{k}|^{2+4/5}}, \tag{10}$$

$$\langle\psi(\mathbf{x})\psi(0)\rangle = const\left[R^{4/5} - |\mathbf{x}|^{4/5}\right], \tag{11}$$

Similarly, if we write a two-point function in the momentum space (10) first, we have to set a cutoff in the k integral to the Fourier transform. To transform it back by cutting off the k-integral at some infrared point, defined by the large scales, $k_{min} \sim 1/r$. This would give (11) Therefore, the physical correlators differ by a term popping up because of the infrared cutoff. It is also well understood that the conformal expression is valid only for the $|\mathbf{k}| << k_{max} \sim 1/a$. This is the UV cutoff setup due to the viscosity. This means we have to use the two-point splitting to make things work. To check the Hopf equation, we need to calculate $\dot{\omega}$ and for that we will use (3), RHS is zero because of the $\nu \to 0$ limit. Therefore we have (12)

$$\dot{\omega} = -e_{\alpha\beta}\partial_\alpha\psi\partial_\beta\partial^2\psi, \tag{12}$$

To calculate the RHS in (12) we have to use the two-point splitting,

$$e_{\alpha\beta}\partial_\alpha\psi(\mathbf{x})\partial_\beta\partial^2\psi(\mathbf{x}) = \lim_{a\longrightarrow 0} e_{\alpha\beta}\partial_\alpha\psi(\mathbf{x}+\mathbf{a}/2)\partial_\beta\partial^2\psi(\mathbf{x}-\mathbf{a}/2), \tag{13}$$

Next, we will use the fusion rule of our primary (14), and then we have to use it to calculate (13). This can be done by taking derivatives on both sides of (15) and searching for the similar order term of $e_{\alpha\beta}\partial_\alpha\psi(\mathbf{x})\partial_\beta\partial^2\psi(\mathbf{x})$.

$$[\psi] \times [\psi] = [\psi] + [I], \tag{14}$$

$$\begin{aligned}\psi(\mathbf{x}+\mathbf{a}/2)\psi(\mathbf{x}-\mathbf{a}/2) &= |a|^{4/5}(I + C_1 a^2 L_{-2} I + ...) \\ &+ |a|^{2/5}(\psi + C_2 L_{-1}\psi + a^2(C_3 L_{-2} + C_4 L_{-1}^2)\psi + ...) + a.h,\end{aligned} \tag{15}$$

Doing the order matching, we get,

$$e_{\alpha\beta}\partial_\alpha\psi(\mathbf{x})\partial_\beta\partial^2\psi(\mathbf{x}) = \lim_{a\longrightarrow 0} const|a|^{2/5}(L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2)\psi, \tag{16}$$

Luckily, in our case, we have the second-level null state in the (2,5) model. Therefore, equation (16) is zero. Hence, the Hopf equation is satisfied. We must consider the influence of viscosity and stirring forces as crucial boundary conditions for our solutions. Currently, we lack a rigorous methodology for their precise implementation, which necessitates us to make educated conjectures. These conjectures should ideally yield solutions that are both more generic and physically realistic than those discussed above.

Constant Flux: Now, as we know, the standard picture that we have of the turbulence involves the Kolmogorov idea of constant flux condition. The condition of matching the perfect fluid and the viscous fluid near the cutoff requires the constant flux condition. In 2D, the relevant flux is the enstrophy flux.

$$H = \int \omega^2 d^2x, \tag{17}$$

Enstrophy is a conserved quantity given by (17) for the perfect fluids.

$$\frac{d}{dt}\langle \omega(\mathbf{x}+\mathbf{a}/2)\omega(\mathbf{x}-\mathbf{a}/2)\rangle = constant, \tag{18}$$

$$[\psi]\times[\psi] = [\phi] + ..., \tag{19}$$

$$\dot{\omega} \sim (L_{-2}\bar{L}_{-1}^2 - \bar{L}_{-2}L_{-1}^2)\phi, \tag{20}$$

The constant flux condition says that the total time derivative of the $< \omega\ \omega >$ is length scale independent. Taking into account the UV cutoff the condition it boils down to (18). Now consider some CFT with the fusion rule (19), and we have already argued above that the $\dot{\omega}$ is given by (20),

$$\langle \dot{\omega}(x+r)\omega(r)\rangle = r^0, \tag{21}$$

$$(\Delta_\phi + 2) + (\Delta_\psi + 1) = 0, \tag{22}$$

The constant flux condition says (21), now from this we get (22). Unlike our last example, we won't demand the second-level degenerate condition. Instead, we need the condition (23), which essentially means the RHS of (19) will vanish under the limit $a \to 0$. This gives $\Delta_\psi < -1$.

$$\Delta_\phi > 2\Delta_\psi, \tag{23}$$

Now we know that the energy spectrum is given by,

$$E(k) \sim k^{4\Delta_\psi + 1}, \tag{24}$$

$$E(k) \sim k^{-3}, \tag{25}$$

This we were expecting since it is steeper than the Kraichan-Kolmogorov approximation (25) because in the numerical simulations, this exponent is found somewhere between -3 to -4. Therefore, the primary with the scaling dimension satisfying the condition (23) and its value is greater than -5/4 will work.

(2,2N+1) Model: We have already seen an example of a non-unitary minimal model, which is the possible field dual of the 2D turbulence. Polyakov, in his paper [1], mentioned that he also does not have a proper classification of the dual CFT, but he gave the most appealing example of (2,2N+1) models. In this model, we have N-primaries.

$$-\Delta_s = \frac{(2N-s)(s-1)}{2(2N+1)} \tag{26}$$

$$[\psi_s]\times[\psi_s] = [\psi_{2s-1}] + [\psi_{2s-3}] + ..., \tag{27}$$

With the scaling dimensions given as (26) and the fusion rule is given as (27) with $2s-1 \le N$. The constant flux condition requires,

$$\Delta_s + \Delta_{2s-1} = -3, \tag{28}$$

$$\Delta_\psi = \Delta_4 = -8/5,\ \Delta_\phi = \Delta_7 = -13/7, \tag{29}$$

$$E(k) \sim k^{-(3+4/7)}, \tag{30}$$

The solution to this equation is (29). The energy spectrum is given as (30). This result is well under the range of the numerical results. Hence, such models are more eligible to be called the Field dual of the 2D turbulence in the inertial range.

Discussions: In the realm of inviscid Hopf equations, a multitude of formal solutions exists. Among these solutions, certain instances emerge where the function ϕ exhibits level-two degeneracy, rendering them more stable than their counterparts. However, the crucial distinction arises when we impose the right boundary conditions in momentum space, a prerequisite for seamless integration with the viscous region. This scenario closely parallels the challenges encountered in laminar flows, wherein delineating the accurate inviscid solution necessitates meticulous consideration of the boundary layer. Just as in the case of laminar flows, here, too, a thorough examination of boundary conditions becomes pivotal to elucidate the correct inviscid solution. In the series of this paper, Yutaka Matsuo, in his paper [3], gave a table of some of the solutions that can possibly be a field dual. This makes this area of research more challenging and interesting to work with.

[1] A. Polyakov, Conformal turbulence, arXiv preprint hep-th/9209046 (1992).

[2] A. S. Monin and A. M. Yaglom, *Statistical fluid mechanics, volume II: mechanics of turbulence*, Vol. 2 (Courier Corporation, 2013).

[3] Y. Matsuo, Some additional solutions of conformal turbulence, Modern Physics Letters A **8**, 619 (1993).