Greens functions contd

L : Differential operators

$$L\,y(x) = f(x) \quad \text{—①}$$

$$L\,G(x,x') = \delta(x-x') \quad \text{—②}$$

solution: $$y(x) = \int_{\in x} dx'\, G(x,x')\, f(x'). \quad \text{—③}$$

$$L\,y(x) = \int_{\in x} dx'\, L\,G(x,x')\, f(x')$$

$$= \int_{\in x} dx'\, \delta(x-x') f(x') = f(x).$$

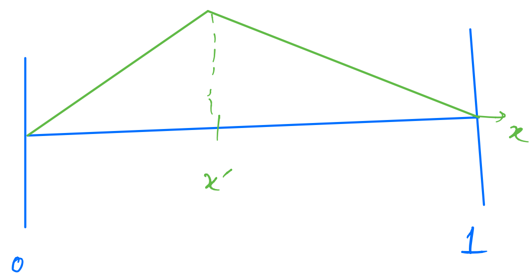
Example

- $L = -\partial_x^2$

$x \in [0,1]$

$y(0) = y(1) = 0$

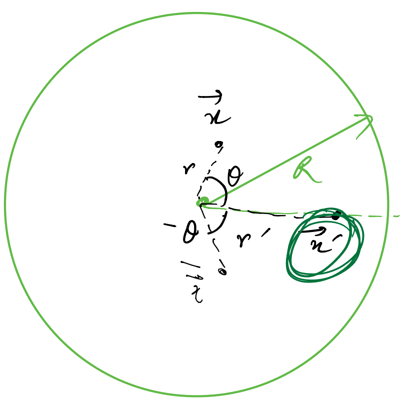
$$G(x,x') = \begin{cases} x(1-x') \,; & x < x' \\ (1-x)\,x' \,; & x > x' \end{cases}$$



- Wave equation in 2D on a disk. : $\nabla^2 u + k^2 u = 0$

$$\boxed{\nabla^2 G + k^2 G = \delta(\vec{x}-\vec{x}')}$$

$$\int_{\in \vec{x}'} d^2x\, \delta(\vec{x}-\vec{x}') = 1$$



$$\boxed{u(\theta) = u(-\theta)}$$

$\Rightarrow$ (No $\sin(m\theta)$ in solution)

$$u(r=R) = 0$$

$\vec{x} : (r,\theta).$

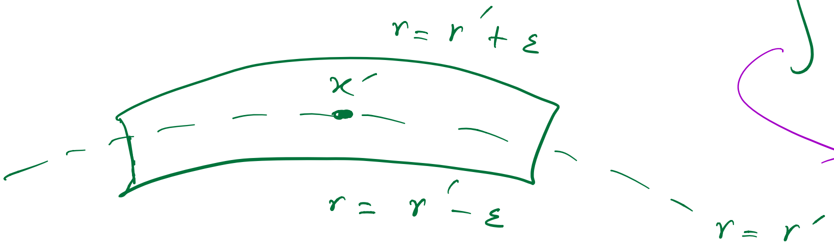
$\vec{x}' \neq \vec{x}$: $\nabla^2 G + k^2 G = 0$

$$G = \sum_m \alpha_m(r) \cos m\theta$$

Integer

$$r < r' : G = \sum_m A_m J_m(kr) \underline{\cos m\theta}$$

$$r > r' : G = \sum_m B_m \big(J_m(kr)\, Y_m(kR) - Y_m(kr)\, J_m(kR)\big) \underline{\cos m\theta}$$



$$\int \nabla^2 G \, d^2x = 1$$

$$\int (\nabla G)_n \, dl$$

$$\int_V \vec{\nabla} \cdot \vec{u} \, d^3x = \int_{\Sigma} \vec{u} \cdot d\vec{S}$$

$$\int_{r'+\varepsilon} \frac{\partial G}{\partial r} dl - \int_{r'-\varepsilon} \frac{\partial G}{\partial r} dl = 1$$

$l = r'\theta$ and thus $dl = r' d\theta$

$$\int d\theta \left(\frac{\partial G}{\partial r}\bigg|_{r'+\varepsilon} - \frac{\partial G}{\partial r}\bigg|_{r'-\varepsilon} \right) = \frac{1}{r'}$$

should contain $\theta = 0$ $(\vec{x}')$

$$\boxed{\left(\frac{\partial G}{\partial r}\bigg|_{r'+\varepsilon} - \frac{\partial G}{\partial r}\bigg|_{r'-\varepsilon} \right) = \frac{1}{r'} \delta(\theta)} = \sum_m C_m \cos m\theta$$

$$\int_{-\pi}^{\pi} \frac{1}{r'} \delta(\theta) \cos m'\theta = \sum_m C_m \int_{-\pi}^{\pi} \cos m\theta \cos m'\theta \, d\theta$$

For $m = m' = 0$ then R.H.S $\quad C_m 2\pi$

For $m = m' \neq 0$ then R.H.S $\quad C_m \pi$

Define: $\varepsilon_m = \begin{cases} 2 & ; m = 0 \\ 1 & ; m \neq 0. \end{cases}$

$$\frac{1}{r'} = \pi C_m \varepsilon_m \Rightarrow C_m = \frac{1}{\pi r' \varepsilon_m}.$$

$$\frac{\partial G}{\partial r}\bigg|_{r'+\varepsilon} - \frac{\partial G}{\partial r}\bigg|_{r'-\varepsilon} = \frac{1}{\pi r'} \sum_m \frac{1}{\varepsilon_m} \cos m\theta \quad \text{— (4)}$$

Continuity of G:

$$\boxed{A_m J_m(kr') = B_m\Big(J_m(kr')Y_m(kR) - Y_m(kr')J_m(kR)\Big)}$$ — (5)

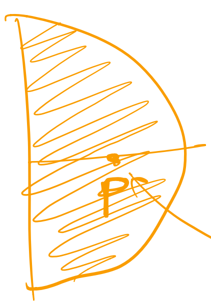
$$\boxed{\begin{aligned}&\text{Eq (4)} \Rightarrow B_m\Big(J_m'(kr')Y_m(kR) - Y_m'(kr')J_m(kR)\Big) \\ &\qquad - A_m J_m'(kr') = \frac{1}{\pi\varepsilon_m k r'}\end{aligned}}$$ — (6)

$$A_m = \frac{J_m(kR)\,Y_m(kr') - J_m(kr')\,Y_m(kR)}{2\,\varepsilon_m J_m(kR)}$$ ✓

$$B_m = \frac{-J_m(kr')}{2\varepsilon_m J_m(kR)}$$ ✓

$$\boxed{J_m(x)Y_m'(x) - J_m'(x)Y_m(x) = \frac{2}{\pi x}}$$

To be done in Tutorial



at $t=0$

T_0

T_1

$$\boxed{\nabla^2 T = \frac{1}{K}\frac{\partial T}{\partial t}}$$

Find $\underline{T(x,y,t)}$ for $t>0$.

INITIAL VALUE PROBLEMS

$$\frac{dy}{dt} - \wp(t)\, y = F(t) \quad \text{with} \quad y(0) = 0$$

$$\left(\frac{d}{dt} - \wp(t)\right) G(t,t') = \delta(t-t')$$

$t > t'$

$$\int \frac{dG(t,t')}{G(t,t')} = \int \wp(t)\, dt$$

$$G = e^{\int_{t'}^{t} \wp(s) ds}$$

$$G(\underline{0}, t') = 0$$

$t < t'$

$$\left(\frac{d}{dt} - \wp\right) G = 0 \quad \text{is} \quad \boxed{G = 0}$$

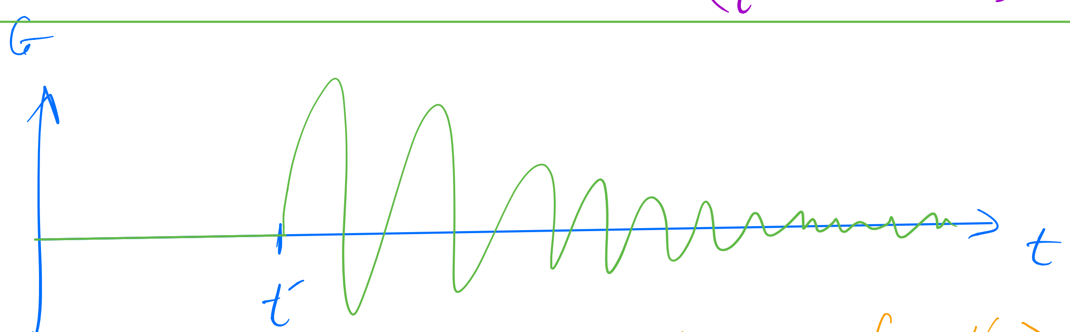
$$\boxed{G(t'+\varepsilon, t') - G(t'-\varepsilon, t') = 1}$$

$t > t'$

$$G(t,t') = \exp\left(\int_{t'}^{t} \wp(s)\, ds\right)$$

Full Solution:

$$G(t,t') = \boxed{\Theta(t-t')} \exp\left(\int_{t'}^{t} \wp(s)\, ds\right).$$



is zero for $t' > t$.

$$y(t) = \int_0^{\infty} G(t,t')\, F(t')\, dt'$$

$$= \int_0^{t} \exp\left(\int_{t'}^{t} \wp(s)\, ds\right) F(t')\, dt'.$$

NEXT CLASS:

2nd ORDER DIFF EQ

Forced, Damped, Harmonic Oscillator:

$$\ddot{x} + 2\gamma\dot{x} + (\Omega^2 + \gamma^2)x = F(t)$$

$$x(0) = \dot{x}(0) = 0.$$