On Socially Optimal Traffic Flow in the Presence of Random Users

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Urban Traffic

Urban traffic can be modelled as a congestion game:

- Commuters are the various players
- The city can be modelled as a graph
- Homes, offices, shopping centers are the **nodes**
- Roads in the city are the **links**
- Cost to traverse each link is a function of the number of commuters using that link

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Traffic Assignment

Traffic assignment refers to:

- Routing city traffic to achieve a *pre-determined equilibrium*
- Examples of some equilibria include:
  - Equal traffic on all roads
  - Preferred use of some roads
  - Biased routing for a group of users
  - Unbiased routing of users

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Socially Optimal Equilibrium

As the central directive, to achieve a socially optimal equilibrium, we:

- Do not discriminate among users
- Discriminate paths only on the basis of their travel costs
- Minimize the sum of travel costs encountered by users, weighted by the number of users facing that cost

Related Work

In all previous works, closed form expressions to model dynamic traffic were used.

However,

- Traffic flow on a path is hardly ever continuous
- Not all users can always be accounted
- Some users follow directives only for a part of their journey
- Unexpected or emergency traffic cannot be handled

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A Glimpse of the Typical Indian Traffic

Figure: Urban Traffic in India [1]

[1]Picture Courtesy of MailOnline
(https://www.dailymail.co.uk/indiahome/indianews/article-3122974)

Typical Traffic Composition

We propose that traffic be classified as:

- Deterministic Traffic
  - Follows central directives or navigation services
  - Includes cars using Google maps, cab services, govt. buses
- Random Traffic
  - Either does not use central directives or does not obey them
  - Includes rickshaws, two-wheelers, unexpected or emergency traffic

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Random Traffic

We assume that random traffic:

- Is present on every link of the network
- Its probability distribution is unknown
- Cannot take arbitrarily large values
- Has zero mean

System Model: The Network

- An urban city $\implies$ Directed Graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$
  where $\mathcal{N}$ is the ordered list of nodes
  $\mathcal{E}$ is the ordered list of edges or links
- For an edge $e \in \mathcal{E}$, let f_e denotes the flow on edge e and $\mathbf{f} = [f_1, f_2, \cdots, f_e, \cdots f_{|\mathcal{E}|}]^T$ be the flow vector of length $|\mathcal{E}|$.
- $f_e = x_e + z_e$
  x_e = Deterministic flow on edge e
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System Model: Traversal Costs

- There is an inherent cost to traverse each link.
- The cost may be a function of travel time, distance, congestion, tension, fuel or a combination of these.
- Let c_e denote the cost to traverse link e.
  We consider c_e to be a function of the **total link flow** f_e only.
- Generally, the cost functions are convex showing the increasing impact of congestion.

  A widely used form is:

$$c_e(f_e) = a_e + b_e \cdot {f_e}^4 \qquad \forall\, e \in \mathcal{E} \tag{1}$$

  where a_e and b_e are constant parameters for every link $e \in \mathcal{E}$.

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Problem Formulation: Objective Function

- Traditionally, for a socially optimal scenario, we tend to minimize:

$$\sum_{\forall\, e \in \mathcal{E}} x_e \cdot c_e(x_e) \tag{2}$$

i.e. minimize the sum of travel costs encountered by users, weighted by the number of users facing that cost.

- Accounting for random traffic, the objective becomes:

$$\min_{x_e} \;\; \mathbb{E}\left[\sum_{\forall e \in \mathcal{E}} f_e \cdot c_e(f_e)\right] \tag{3}$$

- Recall that the central navigation authority can only route the deterministic flow x_e on each link e.

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Problem Formulation: Constraints

- Let $\mathbf{D}$ be a $|\mathcal{N}| \times |\mathcal{N}|$ matrix whose $(i,j)^{th}$ element, D_{ij}, denotes the mean flow of commuters wishing to move from the node i to the node j.
- Let $\mathcal{P}_{mn}$ denotes the set of all open loop paths from the m^{th} node to the n^{th} node.
- Let x_p denote the flow on path p, where $p \in \{\mathcal{P}_{mn}\}$.
- Then the demand constraint is:

$$D_{ij} = \sum_{p \in \mathcal{P}_{ij}} x_p, \qquad \forall\, i,j = 1,2,\ldots,|\mathcal{N}| \tag{4}$$

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- Total deterministic flow on a link:

$$x_e = \sum_{p \ni e} x_p, \qquad \forall\, e \in \mathcal{E}. \tag{5}$$

- Since the flow on any path cannot be negative:

$$x_p \geq 0 \qquad \forall\, p. \tag{6}$$

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- Let z_e denote the random flow on link e.
- z_e is a random variable whose distribution is unknown, but

$$\mathbb{E}[z_e] = 0, \qquad \forall\, e \in \mathcal{E}. \tag{7}$$

- We assume:

$$|z_e| \leq \beta x_e. \tag{8}$$

where β is the spread parameter and $0 \leq \beta \leq 1$.

- Assumption: $\{z_e\}$ is a set of i.i.d. random variables.

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The Optimization Problem

$$\min_{x_e} \ \mathbb{E}\left[\sum_{\forall e \in \mathcal{E}} (x_e + z_e) \cdot c_e(x_e + z_e)\right]$$

subject to:

$$D_{ij} = \sum_{p \in \mathcal{P}_{ij}} x_p \qquad \forall \ i, j = 1, 2, \ldots, |\mathcal{N}|$$

$$x_p \geq 0$$

$$x_e = \sum_{p \ni e} x_p$$

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Proposed Algorithm

- Stochastic constrained optimization problem
- Constraints must be satisfied at every iteration
- Distribution of random variables unknown

- Online solution $\implies$ random variables realized every iteration
- Stochastic Frank-Wolfe Traffic Assignment (SFWTA) Algorithm

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Algorithm SFWTA

1: Choose step sizes ρ_t & γ_t

2: Initialize $t = 0$
Initialize deterministic flow vector $\mathbf{x}$
Initialize cost vector $\mathbf{c}(\mathbf{f}) = \vec{0}$

3: Sample the random flow vector $\mathbf{z}$

4: Randomize $\mathbf{f}$: $\mathbf{f} = \mathbf{x} + \mathbf{z}$

- Choose ρ_t and γ_t such that
  - $\sum_{t=0}^{\infty} \rho_t = \infty$
  - $\sum_{t=0}^{\infty} \rho_t^2 < \infty$
  - $\sum_{t=0}^{\infty} \gamma_t = \infty$
  - $\sum_{t=0}^{\infty} (\gamma_t^2 / \rho_t) < \infty$
- Initialize deterministic flow vector $\mathbf{x}$ using **Dijkstra's algorithm**
- Sample each random variable z_e independently
- Find total flow on each link

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1: Compute $\mathbf{c}'$ where
$c'_e = \nabla F(x_e, z_e)$
Update cost vector $\mathbf{c}$ s.t.
$\mathbf{c} = (1 - \rho_t)\mathbf{c} + (\rho_t)\mathbf{c}'$

2: Compute shortest paths flow $\mathbf{y}$ with link costs as c_e above

3: $\mathbf{x} = (1 - \gamma_{t+1})\mathbf{x} + (\gamma_{t+1})\mathbf{y}$

4: Iterate till convergence

- $F(x_e, z_e) = f_e \cdot (a_e + b_e \cdot f_e^4) = (x_e + z_e) \cdot (a_e + b_e \cdot (x_e + z_e)^4)$
- Cost to traverse each link = Stochastic gradient of objective function
- Projection taking step converted to $|\mathcal{N}|^2$ shortest path problems
- Convergence gives deterministic flow for which expected social cost is least

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Numerical Results: Dummy Network

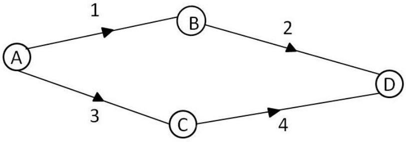

Figure: Model Traffic Network

Single Demand Model:
Unity flow from A (node 1) to D (node 4)

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{a} = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.5 \\ 0.5 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0.6 \\ 0.6 \\ 0.1 \\ 0.1 \end{bmatrix}.$$

Assume $z_e \sim U[-x_e, x_e]$

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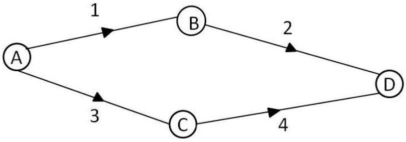

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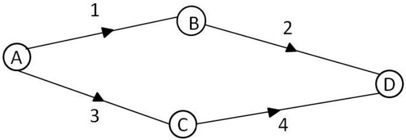

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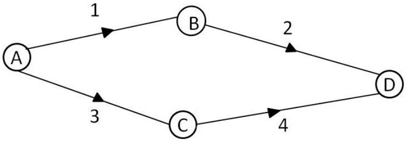

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Numerical Results: Convergence

Without random traffic, optimal deterministic flow is:

$$\mathbf{x} = \begin{bmatrix} 0.5238 \\ 0.5238 \\ 0.4762 \\ 0.4762 \end{bmatrix}$$

Accounting for uncertainties, optimal flow becomes:

$$\mathbf{x} = \begin{bmatrix} 0.4206 \\ 0.4206 \\ 0.5794 \\ 0.5794 \end{bmatrix}$$

for $z_e \sim \mathrm{U}[-x_e, x_e] \implies \beta = 1$

Numerical Results: Convergence

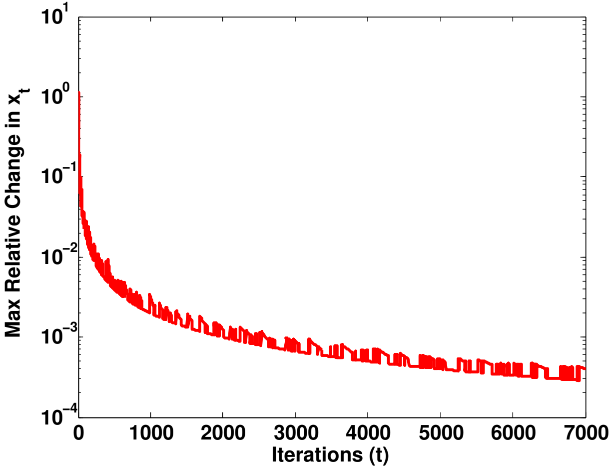

Figure: Max. relative change in flow at any iteration with iterations

Numerical Results: Running Mean of Costs

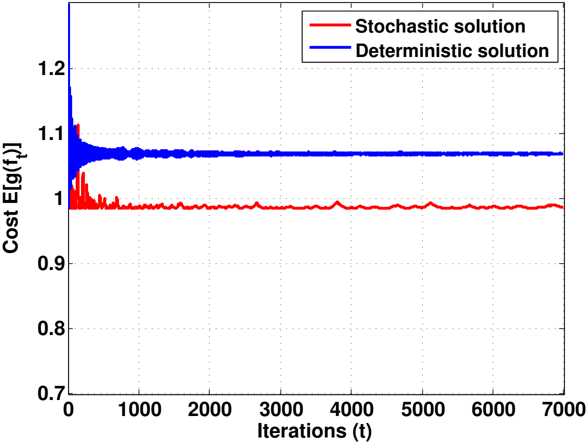

Figure: Running mean of the costs of the stochastic & deterministic solution strategies when $\beta = 1$

Numerical Results: Impact of Spread Parameter

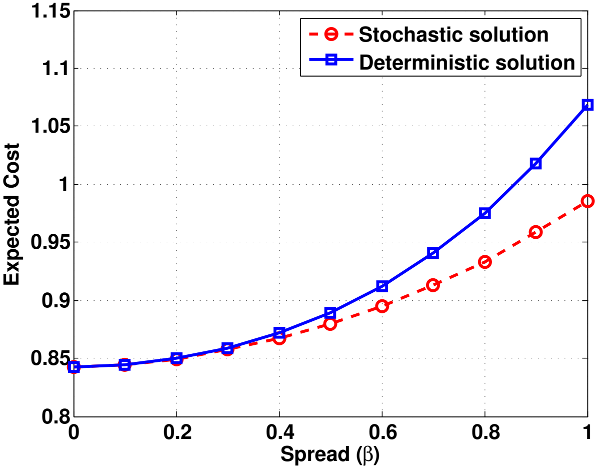

Figure: Stochastic solution strategy performs increasingly better than deterministic solution strategy as β increases

Conclusion & Future Work

We proposed:

- A framework to determine socially optimal flow in stochastic environments, and
- An online variant of the stochastic Frank-Wolfe algorithm to find the deterministic flow which gives least average cost in a stochastic environment

Possible future work:

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Thank You!