

# Modeling and Analysis of Heterogeneous Traffic Networks with Anarchists and Socialist Traffic

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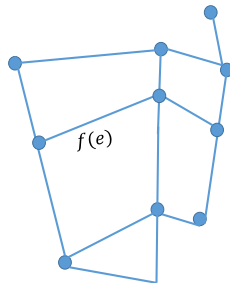
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# Traffic Assignment

- Traffic planning of any urban city or communication networks
- Understand the traffic flow in city roads
- A road network  $N$ : a graph  $\mathcal{G}$  with nodes  $V$  and edges  $E$
- Each link (road)  $e \in E$ 
  - Has flow of  $f_e$
  - The cost of this link (e.g. time taken by a user to cross this link)

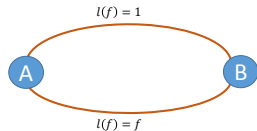
$$c_e = l_e(f_e).$$

- Cost depends on the congestion in the link!!
- Latency function  $l_e$  depends on the link characteristics, for example, type of the road, its capacity, construction materials.



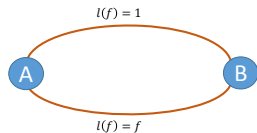
## Two Link Example

- Consider an example with two links between two cities A and B
- First link is a highway: constant travel time of 1 hour
- Second link is a street: travel time linear with congestion
- Total flow is 1
- Which route will people take?



## Two Link Example: Nash Equilibrium

- Let us consider Nash Equilibrium first
- Every person tries to minimize his time- choose a path with the least cost (Anarchist behavior)
- Second one as it is always less than or equal to 1 hour
- If every one does the same, the cost incurred to each person will be 1 hour.
- Total cost is 1 unit of flow  $\times$  1 hour for each person=1
- This is also called Wardrop Equilibrium - Paths with the least cost are taken by users.



Which route will people take?

## Two Link Example: Social Optimum

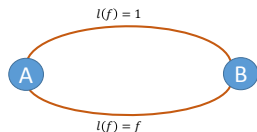
- Consider society perspective: Reduce total cost
- Persons listen to central directive (Socialist behavior)
- If the flows in two paths are  $f_1$  and  $f_2$ , total cost is

$$c = f_1 \times 1 + f_2 \times f_2$$

- Total flow is 1:  $f_1 + f_2 = 1$
- The solution minimizing the total cost is

$$f_1 = f_2 = 0.5$$

- Total cost =  $\frac{3}{4}$  which is better than NE



If citizens don't listen to the central directive, they have to suffer  $4/3$  times more cost

Termed price of Anarchy!

# NE and SO for General Network

- Let vector  $x = \{x_e, e \in \text{edges}\}$  denote flow in all edges.
- $x$  is the NE flow if it solves (Wardrop Equilibrium)<sup>1</sup>

$$x = \arg \min \sum_{e \in E} \int_0^{x_e} l_e(z) dz$$

- $x$  is the SO flow if it solves

$$x = \arg \min \sum_{e \in E} x_e l_e(z)$$

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<sup>1</sup>Wardrop1952.

# Optimal Traffic Assignment

- User's perspective on how to choose optimal path may be different
  - Users may follow route recommendations from a navigation system
  - Users may follow the least costly path
  - Users may follow fixed/random paths
- Past literature has focus on solving games with single type of users
  - NE: as a solution of an optimization problem using Wardrop's equilibrium<sup>2</sup> .
  - Social Optimum: by solving an optimization problem minimizing the total cost.
- What about a traffic network with different type of users
  - Network with socialists and random users<sup>3</sup>

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<sup>2</sup>Wardrop1952.

<sup>3</sup>Gupta2008.

- HetGame: a game among heterogeneous users with different path selection objective.
- The dynamics and equilibria for insights in planning and user routing
- We focus on two types of users:
  - Anarchists who decide their route by choosing the least costly available path: Nash Equilibrium
  - Socialist behavior who chose paths minimizing the average cost of everyone: City Planner's Dream

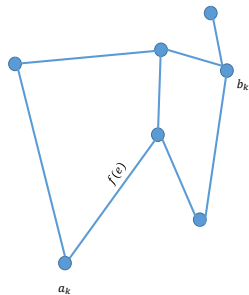


# Contributions

- We consider a heterogeneous traffic network with multiple users classes with different path selection objectives
- Focus on two classes of users: anarchists and socialists
- A framework to derive optimal/equilibrium flow in such a heterogeneous game
- Proposed two new metrics: price of  $\alpha$  anarchy and price of good behavior
- Insights about these systems to help formulate central directives to make the social optimal solution to be equal to the equilibrium

# System Model: Traffic Network

- The traffic network  $N$ : a graph  $\mathcal{G}$  with nodes  $V$  and edges  $E$
- $K$  source destination pairs  $\{p_k : (a_k, b_k), k = 1 : K\}$



# System Model

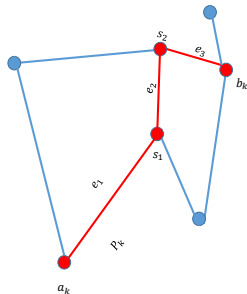
- For  $k$ th pair:
  - $d_k$ : required flow between source and destination
  - A path  $P_k$  consists of a set of connected edges *i.e.*

$$P_k = \{e_1, e_2, \dots, e_n\} :$$

$$e_1 = (a_k, s_1), e_2 = (s_2, s_3), \dots, e_n = (s_n, b_k),$$

- $\mathcal{P}_k = \{P_k\}$  denote the set of all such path
- Set of all paths  $\mathcal{P} = \cup_{k=1:K} \mathcal{P}_k$ .
- Let the flow in each path  $P \in \mathcal{P}$  be  $f_P$ .

For any edge  $e \in E$ , the total traffic flow in the link is  $f_e = \sum_{P \ni e} f_P$ .



## System Model: User Classes

- Two types of users: Anarchists and Socialists with ratio  $\alpha_1 : \alpha_2 = \alpha : 1 - \alpha$
- Required flow between any source destination pair consists of the two users in the same ratio
- Strategy of the type  $m$  is the vector  $X_m = (X_{mP})_{P \in \mathcal{P}}$  (consisting of flow in each path)
  - Given the strategy, the flow of type  $m$  users in a path  $P$  is given as  $\alpha_m X_{mP}$
- Given strategy, the total flow in any link  $e$

$$f_e = \sum_{P \ni e} \sum_{m \in \mathcal{M}} \alpha_m X_{m,P}.$$

## $\alpha$ -Anarchy HetGame

- HetGame with two classes of users: socialists with  $\alpha_s = (1 - \alpha)$  proportion and anarchists with  $\alpha_a = \alpha$  proportion.
- For each source and destination pair  $p_k$ ,
  - Anarchist flow is  $\alpha d_k$  and
  - Socialist flow is  $(1 - \alpha)d_k$ .
- $X_s = x$ : socialist strategy  
 $X_a = y$ : anarchist strategy.
- Total flow in any edge  $e$ :

$$f_e = \sum_{P \ni e, P \in \mathcal{P}} [(1 - \alpha)x_P + \alpha y_P].$$

# Constraints

For the  $k$ th pair

$\mathcal{S}_s$  : sum of the socialist flows in all the paths = the total demand flow of socialists

$$\sum_{P_k \in \mathcal{P}_k} (1 - \alpha)x_{P_k} = (1 - \alpha)d_k,$$
$$\implies \sum_{P_k \in \mathcal{P}_k} x_{P_k} = d_k,$$

Similarly, for the anarchist, the flow constraint is given as

$$\mathcal{S}_a : \sum_{P_k \in \mathcal{P}_k} y_{P_k} = d_k.$$

# Joint Flow

- In the game where both classes co-exist, each class will try to optimize their own flow in presence of the flow of other class according to their own path selection strategy
- Framework to derive the joint optimal flow for the users of the two classes
- Two step solution
  - Anarchists will try to achieve nash equilibrium (NE) given the socialist flow
  - Socialists must find a socialist flow so that the responding NE strategy from the anarchists achieves the minimum social cost

## Step 1: NE Given Socialist Flow $x$

### Theorem (Modified Wardrop Equations)

*Given socialist strategy  $x$  in any general  $\alpha$ -anarchy HetGame, the NE of anarchist users is given as the solution of:*

$$y^* = \arg \min_y \sum_e \int_0^{y_e} l_e \left( \sum_{P \ni e} (1 - \alpha)x_P + \alpha z \right) dz$$

such that  $y_e = \sum_{P \ni e} y_P, \sum_{P_k \in \mathcal{P}_k} y_{P_k} = d_k.$



## Step 2: Optimal Socialist Flow

Socialist users must choose a strategy  $x^*$  that minimizes the total cost of the network

$$x^* = \arg \min_x \sum_e f_e l_e(f_e)$$

$$\text{such that } f_e = \sum_{P \ni e} (1 - \alpha) x_P + \alpha y_P^*(x),$$

$$\sum_{P_k} x_{P_k} = d_k$$

# Optimal Flow

Optimal strategy  $(x^*, y^*)$ : simultaneous solution of following sub-problems  $\mathcal{S}_1, \mathcal{S}_2$ :

$$\mathcal{S}_1 : \quad x^* = \arg \min_x \sum_e f_e l_e(f_e)$$

$$\mathcal{S}_2 : \quad y^*(x) = \arg \min_y \sum_e l'_e(y_e, x)$$

$$\text{such that } f_e = \sum_{P \ni e} (1 - \alpha)x_P + \alpha y_P^*(x),$$

$$\text{such that } y_e = \sum_{P \ni e} y_P,$$

$$\sum_{P_k} x_{P_k} = d_k$$

$$\sum_{P_k} y_{P_k} = d_k.$$

where  $l'_e(y_e, x)$  is the modified link cost function for anarchist

$$l'_e(y_e, x) = \int_0^{y_e} l_e \left( \sum_{P \ni e} (1 - \alpha)x_P + \alpha z \right) dz.$$

## Price of Anarchy and Good Behavior

**Price of  $\alpha$ -anarchy:** Relative increase in the average cost due to presence of  $\alpha$  proportion of anarchists *i.e.*

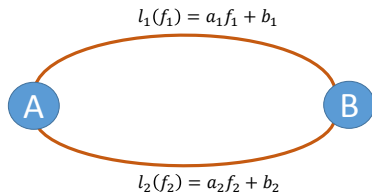
$$P_A = \frac{\text{Total cost with } \alpha \text{ anarchists and } (1 - \alpha) \text{ socialists}}{\text{Total cost of system with no anarchist}}$$

**Price of good behavior:** Relative cost of following central directive compared to that when being selfish *i.e.*

$$P_G = \frac{\text{Average cost of a socialist user}}{\text{Average cost of an anarchist user}}.$$

## Example: General Two-link Linear Network

- General two link network with linear latency
- Two possible paths in the network
- Link  $i$  ( $i = 1, 2$ ) has latency  $a_i f_i + b_i$  where  $f_i$  is the flow in that link
- Fraction of anarchists:  $\alpha$



A two node network.  $b_1 > b_2$ . The required demand flow is 1 between node A and B.

## Example: General Two-link Linear Network

- In the absence of socialists, the equilibrium flow is given by NE as

$$f_1 = 1 - A$$

$$f_2 = A \text{ where}$$

$$A = \frac{b_1 - b_2 + a_1}{(a_1 + a_2)}$$

- In the absence of the anarchist traffic, the social optimal solution is given as

$$f_1 = 1 - f_{2\text{opt}}$$

$$f_2 = f_{2\text{opt}} \text{ where}$$

$$f_{2\text{opt}} = \frac{a_1 + (b_1 - b_2)/2}{a_1 + a_2}$$

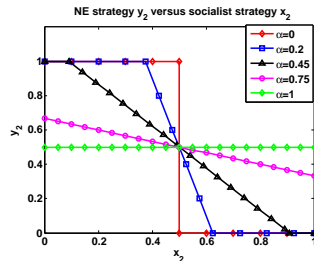
For simplicity, we assume  $0 \leq f_{2\text{opt}} \leq A \leq 1$

## Example: General Two-link Linear Network

- Anarchist strategy:  $(y_1, y_2)$   
Socialist strategy:  $(x_1, x_2)$ .
- NE of the anarchist users given  $x_2$  as

$$y_2(x_2) = \begin{cases} 1 & \text{if } R_1 : x_2 \leq \frac{A-\alpha}{1-\alpha} \\ \frac{A}{\alpha} - \frac{1-\alpha}{\alpha} x_2 & \text{if } R_2 : \frac{A}{1-\alpha} \geq x_2 \geq \frac{A-\alpha}{1-\alpha} \\ 0 & \text{if } R_3 : x_2 \geq \frac{A}{1-\alpha} \end{cases}$$

- Socialist can indirectly force anarchist to choose an arbitrary strategy via a well designed socialist flow
- Control can be limited for particular values of  $\alpha$ , e.g.  $\alpha < A$  &  $\alpha > 1 - A$

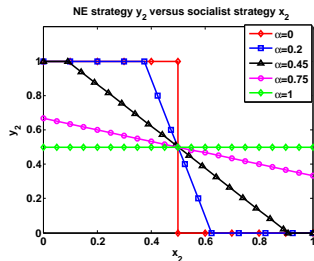


$$a_1 = 0.3, b_1 = 1.0$$

$$a_2 = 0.7, b_2 = 0.8.$$

# Socialist can indirectly force anarchists!

- Pure Equilibrium
  - Anarchist equilibrium is  $(1 - A, A) = (.5, .5)$
  - Optimal social flow is  $(0.6, f_{2opt} = 0.4)$
- For  $\alpha = 0.2$ ,  $y_2$  shifts from the value 1 in  $R_1 = (0, 0.375)$  to the value 0 in  $R_3 = (0.625, 1)$ .
  - Diverting 20% socialist traffic to the second path, all anarchists can be forced to take the second path
  - by diverting 70% socialists to the second path, all anarchists can be forced to take the first path.

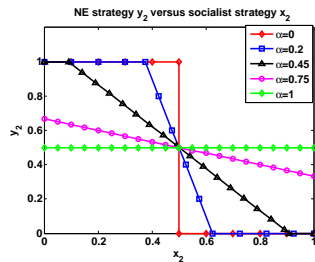


$$a_1 = 0.3, b_1 = 1.0$$

$$a_2 = 0.7, b_2 = 0.8.$$

# Control on flow can be limited

- Not all of the above regions exist for particular values of  $\alpha$ 
  - $R_1$  doesn't exist for  $\alpha < A$  and  $R_2$  doesn't exist for  $\alpha > 1 - A$ .
  - As  $\alpha$  increases, the impact of  $x_2$  on  $y_2$  decreases
- Restriction on the fraction of anarchists affectable by the central planner.
- For example, for  $\alpha = 0.75$ , only  $A/\alpha = 66.67\%$  anarchists at max can be forced to take the second path.



$$a_1 = 0.3, b_1 = 1.0$$
$$a_2 = 0.7, b_2 = 0.8.$$

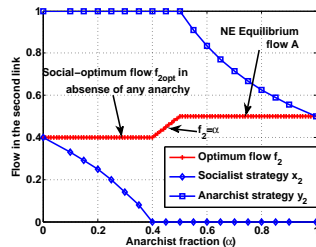


# Optimal Flow

1. When  $\alpha \leq f_{2\text{opt}}$ , the optimal strategy for socialist and anarchist is

$$(x_1, x_2) = \left( \frac{1 - f_{2\text{opt}}}{1 - \alpha}, \frac{f_{2\text{opt}} - \alpha}{1 - \alpha} \right)$$
$$(y_1, y_2) = (0, 1).$$

Total flow in two paths is the social optimum flow as the socialists are able to compensate for the anarchist flow and bring the system to the social optimum.



Equilibrium flow

$$a_1 = 0.3, a_2 = 0.7, b_1 = 1, b_2 = 0.8.$$

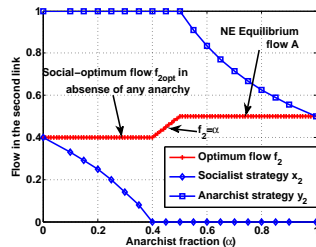
# Optimal Flow

2. When  $f_{2\text{opt}} < \alpha \leq A$ , the optimal strategy is

$$(x_1, x_2) = (1, 0) \quad (y_1, y_2) = (0, 1).$$

Socialist cannot compensate for the anarchist traffic.  
All anarchists take the second link and all socialists take the first link.

The total flow in the network is  $(f_1, f_2) = (1 - \alpha, \alpha)$ .  
Flow in second path increases until system reaches the pure anarchy (Nash equilibrium) state.

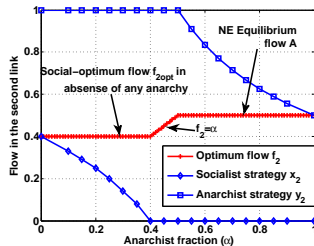


Equilibrium flow

$$a_1 = 0.3, a_2 = 0.7, b_1 = 1, b_2 = 0.8.$$

# Optimal Flow

3. When  $\alpha > A$ , all strategies are optimum.  
The total flow is constant at  $A$  (NE)  
The two links offer the same cost of travel.

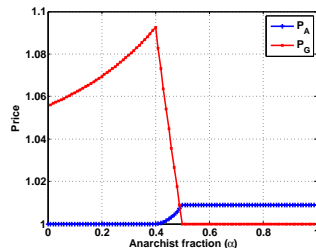


Equilibrium flow

$$a_1 = 0.3, a_2 = 0.7, b_1 = 1, b_2 = 0.8.$$

# Prices

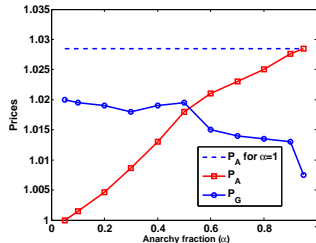
- Increasing anarchy will hurt anarchists
- With increase in fraction of anarchist, price of being a good citizen increases,
- But after a threshold, it starts decreasing and eventually becomes equal to 1 where all users start seeing the NE cost in both paths.



Price of  $\alpha$  anarchy and price of good behavior with varying  $\alpha$   
 $a_1 = 0.3, b_1 = 1.0$   
 $a_2 = 0.7, b_2 = 0.8$

## Example: Real World Traffic Network

- 24-node road network in Sioux Falls, SD with 76 links connecting these nodes and 528 origin-destination pairs
- $l_e(x_e) = d \left(1 + b \left(\frac{x_e}{c}\right)^a\right)$ .
- For every pair, four shortest paths are considered
- Alternate minimization
- $P_A$  increases with  $\alpha$ : Increasing anarchy will hurt anarchists also.
- $P_G$  decreases with  $\alpha$ : Motivation for more people to be socialists and follower of central directives.



# Conclusion

## Summary:

- Derived framework for general traffic network with heterogeneous users, including an  $\alpha$ -anarchy
- Discussion on how social flow can be used to affect the flow of anarchists in desired direction

## Future work:

- HetGame with more number of user classes (e.g., proportionally fair, random, and fixed path-followers)
- Analysis in the presence of random noise
- Socialists with objective to minimize the average socialist cost, neglecting the anarchist portion of the full social cost



# Algorithm for Optimal Flow

- The link cost function  $l_e(\cdot)$  is generally convex
- e.g., of the form  
 $l_e(x_e) = b_e + a_e(x_e/c_e)^d, d \geq 1.$
- The above optimization is a convex problem for convex link cost functions
- Can be solved using the following alternative minimization of  $\mathcal{S}_1$  given  $y^*$  and  $\mathcal{S}_2$  given  $x^*$ :

## Alternative Minimization:

Initialize to  $x_0^*, y_0^*, i = 0.$

Solve optimization  $\mathcal{S}_1$  to compute  $x_1^*, y_0^*.$

Solve optimization  $\mathcal{S}_2$  to compute  $x_1^*, y_1^*.$

**while** Change in solution is greater than tolerance **do**

At step  $i,$

$$x_{i+1}^*, y_i^* = \mathcal{S}_1(x_i^*, y_i^*)$$

$$x_{i+1}^*, y_{i+1}^* = \mathcal{S}_2(x_{i+1}^*, y_i^*)$$

$i \rightarrow i + 1.$

**end while**



## Proof for Theorem

Given the socialist strategy  $x$ , the anarchist will decide their flow by computing the NE for their flow  $\alpha y$ .

Given  $x$ , the latency they face in any edge  $e$  is given by  $L_e(\sum_{P \ni e} \alpha y_P)$  where  $L(w) = l_e(\sum_{P \ni e} (1 - \alpha)x_P + \alpha w)$  for any strategy  $y$ .

Let us assume  $y'_e = \sum_{P \ni e} \alpha y_P$ . With these new latency function, the NE of the flow following anarchist strategy will be equal to Wardrop equilibrium which is given as the solution of the following problem

$$\min_y \sum_e \int_0^{y'_e} L_e(w) dw$$

$$\text{such that } y'_e = \sum_{P \ni e} \alpha y_P, \sum_{P_k} y_{P_k} = d_k$$

which is equivalent to

$$\min \sum_e \int_0^{y'_e} l_e(\sum_{P \ni e} (1 - \alpha)x_P + w) dw$$

$$\text{such that } y'_e = \sum_{P \ni e} \alpha y_P, \sum_{P_k} y_{P_k} = d_k$$

which will give the desired result with substitution  $y'_e = \alpha y_e$  and  $z = \alpha w$ .