Department of Mathematics
Indian Institute of Technology Kanpur
MTH101R, 2nd Mid Term Test
March 8, 2011

Marks: 40
Time: 1 Hour

***Answer all questions. A*ll the parts of each question must be answered in continuation; otherwise they will not be graded*.**

1. (a) Show that there is no continuous $f(x)$ such that $\int_0^1 x^n f(x)\, dx = \frac{1}{\sqrt{n}}$ for all natural numbers n. [5]

Solution: Suppose such a function exists. Let $\sup_{x\in[0,1]} f(x) = M$. Then,

$$\frac{1}{\sqrt{n}} = \left|\int_0^1 f(x)x^n dx\right| \le M\left|\int_0^1 x^n dx\right| = \frac{M}{n+1}$$ [3 marks]

$\Rightarrow 1 \le \frac{\sqrt{n}M}{n+1} \to 0 \ \ as\ n \to \infty \Rightarrow$ a contradiction. [2 marks]

(b) Determine the limits (i) $\lim_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}$ and (ii) $\int_0^1 \lim_{n\to\infty}\frac{ny^{n-1}}{1+y}$. [5]

Solution: (i) Integrating by parts, $\lim_{n\to\infty}\int_0^1 \frac{d(y^n)}{1+y} = \frac{1}{2} + \lim_{n\to\infty}\int_0^1 \frac{y^n}{(1+y)^2}dy$ [1 mark]

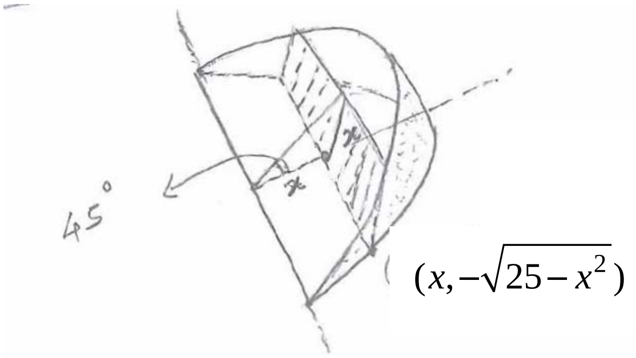
and $\int_0^1 \frac{y^n}{(1+y)^2}dy \le \int_0^1 y^n dy = \frac{1}{n+1} \to 0$ [2 marks]

(ii) For $0 < y < 1,\ \ln(ny^{n-1}) = (n-1)[\ln y + \frac{\ln n}{n-1}] \to -\infty,\ as\ n \to \infty$ [2 marks]

$\Rightarrow \lim_{n\to\infty}\frac{ny^{n-1}}{(1+y)} = 0\ \ for\ \ 0 < y < 1.$

2. (a) A curved wedge is cut from a cylinder of radius 5 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at 45° angle at the center of the cylinder. Find the volume of the wedge. [5]

Solution: Each cross section is a rectangle of area $A(x) = x \times 2\sqrt{25 - x^2}$. [3 marks]



Therefore, Required Volume = $\int_0^5 2x\,\sqrt{25-x^2}\,dx = \dfrac{250}{3}$ [2marks]

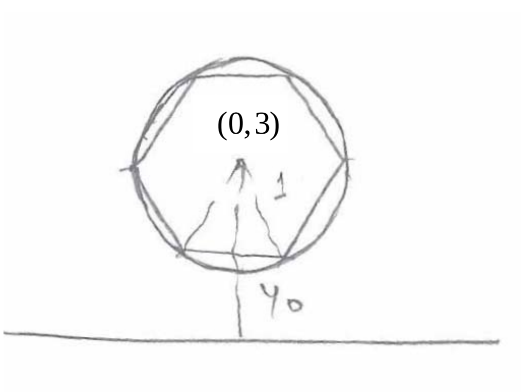
(b) A regular hexagon is inscribed in the circle $x^2 + (y-3)^2 = 1$ and is rotated about $x - axis$. Using Pappus Theorem, find the volume and surface area of the solid so formed. [5]

Solution: Area A of the hexagon is $\dfrac{3\sqrt{3}}{2}$ and the perimeter L is 6. [2 mark]

The distance ρ of centroid from the axis of rotation is 3. [1 mark]

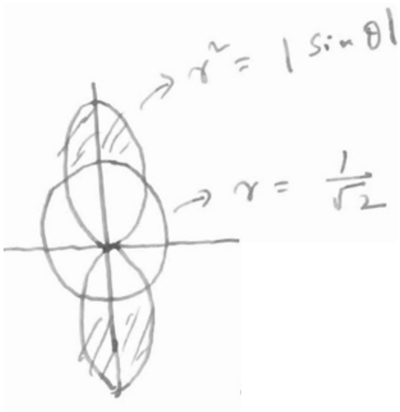
Therefore, by Pappus Theorem, generated volume $V = 2\pi\,\rho\,A = 9\sqrt{3}\,\pi$ [1 mark]

generated surface area $S = 2\pi\,\rho\,L = 36\pi$ [1 mark]



3. (a) Sketch the region D that is bounded by the curve $r^2 = |\sin\theta|$ but is not included in the circle $r = \frac{1}{\sqrt{2}}$.
Determine the area of region D. [5]

Solution:
Correct Sketch: [2 marks]



Point of intersection in the first quadrant: $\sin\theta = 1/2 \Rightarrow \theta = \pi/6$ [1 mark]

Required Area = $4\int_{\pi/6}^{\pi/2} \frac{1}{2}(r^2 - \frac{1}{2})\,d\theta = 2\int_{\pi/6}^{\pi/2} (\sin\theta - \frac{1}{2})\,d\theta$

$= 2[(\cos\frac{\pi}{6} - \cos\frac{\pi}{2}) - (\frac{\pi}{4} - \frac{\pi}{12})] = \sqrt{3} - \frac{\pi}{3}$ [2 marks}

(b) Using the parametric form of the equation of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, find the surface area generated by revolving this ellipse about $x-axis$. Leave your result in the form of the definite integral, do not evaluate it. [5]

Solution: Required Surface Area of Revolution =
2 times the Surface Area of revolution of the part of ellipse in the first quadrant
$= 2\int 2\pi y\,ds$

$= 4\pi \int_0^{\pi/2} (2\sin\varphi)\sqrt{(\frac{dx}{d\varphi})^2 + (\frac{dy}{d\varphi})^2}\,d\varphi$, (*) [2 marks]

The parametric equation for the part of ellipse in the first quadrant,

$x = 3\cos\varphi,\ y = 2\sin\varphi,\ 0 \le \varphi \le \frac{\pi}{2}$. [1 mark]

$\Rightarrow (*) = 8\pi \int_0^{\pi/2} (\sin\varphi)\sqrt{9\sin^2\varphi + 4\cos^2\varphi}\,d\varphi$ [2 marks]

4. (a) Let the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for some x_0. Prove that it converges for all x such that $|x| < |x_0|$. [5]

Solution: $\left|a_n x^n\right| = \left|a_n x_0^n\right| \left|\frac{x}{x_0}\right|^n \le M \left|\frac{x}{x_0}\right|^n, \; for \; all \; n.$ [3 marks]

The result follows by comparison test. [2 marks]

(b) Test the convergence of improper integral $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$. [5]

Solution: Consider the integral $-\int_0^1 \frac{\ln x}{\sqrt{x}} dx$ [1 mark]

Use Limit Comparison Test with $\frac{1}{x^p}$, where $\frac{1}{2} < p < 1$, [1 mark]

$\lim_{x \to 0} \frac{-\ln x}{x^{(1/2)-p}} = \lim_{x \to 0} \frac{1}{p-(1/2)} x^{p-(1/2)} = 0, \; for \; p > \frac{1}{2}$ [2 marks]

and

$\int_0^1 \frac{1}{x^p} dx$ Converges, for $p < 1$. [1 marks]

$\Rightarrow$ Given Integral converges.