Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MSO202 Assignment 1
Introduction To Complex Analysis

*The problems marked (**T**) need an explicit discussion in the tutorial class. Other problems are for enhanced practice whose hints for solutions will be provided on course web-page..*

1. Sketch the following sets and determine which ones of these are domains:
(a) $|z-2+i|<1$ (b) $S=\{z:|z-1|<1 \; or \; |z+1|<1$ **(T)**(c) $0 \le \arg z \le \pi/4$
(T)(d) $|z-4| \ge |z|$ (e) $|\operatorname{Re} z| > a > 0$ (f) $|\operatorname{Im} z| \le |\operatorname{Re} z|$ **(T)**(g) $|z+ia|<|z-a|$ *for* $a>0$.

2. Which of the following functions f(z) can be defined at z = 0 so that they become continuous at z = 0:
(T)(a) $2z\dfrac{\operatorname{Re} z}{|z|}$ (b) $\dfrac{\operatorname{Re}(z^2)}{|2z|^2}$ **(T)**(c) $\dfrac{3\operatorname{Re} z}{z}$ (d) $\dfrac{iz}{|z|}$ (e) $\dfrac{(\operatorname{Re} z)^2 \operatorname{Im} z}{(\operatorname{Re} z)^4+(\operatorname{Im} z)^2}$

3. Show that, for $f(z)=\dfrac{[(1-i)z+(1+i)\bar{z}]^2}{z\,\bar{z}}$,
$\lim\limits_{x\to 0}[\lim\limits_{y\to 0} f(z)] = \lim\limits_{y\to 0}[\lim\limits_{x\to 0} f(z)]$ but $\lim\limits_{z\to 0} f(z)$ does not exist.

4. Show that (a) *f(z) = Re z* is not differentiable for any z (b) $f(z)=|z|^2$ is differentiable only at z = 0.

5. **(T)**Show that the function

$$f(z)=\begin{cases} \dfrac{z^5}{|z|^4} & if \;\; z \ne 0 \\ 0 & if \;\; z=0 \end{cases}$$

is continuous at z = 0, fiirst order partial derivatives of its real and imaginary part exist at z = 0, but f(z) is not differentiable at z = 0.

6. Prove that if a function f(z) is differentiable at z = 0, it is continuous at z = 0.

7. Show that for each of the following functions Cauchy-Riemann equations are satisfied at the origin. Also determine whether these functions are differentiable at z = 0. Are these functions analytic at z = 0?

(T) (i) $f(z)=\sqrt{|\operatorname{Re}(z)\operatorname{Im}(z)|}$ (ii) $f(z)=xy^2+i\,yx^2$, $z=x+i\,y$ (iii) $f(z)=\begin{cases} \dfrac{\operatorname{Im}(z)^2}{|z|^2} & if \;\; z \ne 0 \\ 0 & if \;\; z=0 \end{cases}$

8. Find the domain in which the function
$f(z)=|\operatorname{Re} z^2|+i|\operatorname{Im} z^2|$
is analytic.

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9. **(T)**Show that the derivative of a real valued function $f(z)$ of a complex variable z, at any point, is either zero or it does not exist.

10. **(T)**Prove that

(a) If $f(z)$ *and* $\overline{f(z)}$ both are analytic in a domain D, then $f(z)$ is a constant function in D.
(b) If $f(z)$ is analytic and $f'(z) \equiv 0$ in a domaind D, then $f(z)$ is a constant function in D.
(c) If $f(z)$ is analytic in a domain D and $u_x + v_y = 0$ in D, then $f'(z)$ is constant in D.

11. **(T)**Let $f(z) = u + i\,v = R\,e^{i\varphi}$ be an analytic function in a domain D. Prove that if any of the functions u, v, R, φ is identically constant in D, then f(z) is a constant function in D.

12. If f(z) is an analytic function in a domain D, prove that

$$\nabla^2 |f(z)|^2 = 4|f'(z)|^2 .$$

13. Using CR equations in cartesian coordinates, obtain the following CR equations in the polar coordinates: $r\,u_r = v_\theta$, $r\,v_r = -u_\theta$. Express $f'(z)$ in terms of the partial derivatives with respect to r *and* θ .