Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MSO202 Assignment 2
Introduction To Complex Analysis

The problems marked (T) need an explicit discussion in the tutorial class. Other problems are for enhanced practice.

Note: For uniformity, use *ln x* for natural logarithm of real variable x, l*og z* for logarithmic function of complex variable z and *Log z* as the Principal Branch of *log z*.

1. **(T)** Show that if $\operatorname{Re} z_1 > 0$ *and* $\operatorname{Re} z_2 > 0$, *then* $Log(z_1 z_2) = Log(z_1) + Log(z_2)$.
2. Express the following complex numbers in the form a + i b:

   $(i)\ \log(Log\, i)\quad (ii)\ \sinh(e^{i})$ **(T)** $(iii)\ (-3)^{\sqrt{2}}\quad (iv)\ 1^{-i}$
3. **(T)** Prove that $(a)\ |\sinh(\operatorname{Im} z)| \le |\sin(z)|\ (b)\ |\cos(z)| \le \cosh(\operatorname{Im} z)$. Deduce that $|\sin z|$ *and* $|\cos z|$ tend to ∞ *as* $z \to \infty$ in either of the angles $\delta \le \arg z \le \pi - \delta,\ \pi + \delta < \arg z < 2\pi - \delta,$ *where* $0 < \delta < \pi/2$. (b) Find the points on the square region $-\pi \le \operatorname{Re} z \le \pi,\ -\pi \le \operatorname{Im} z \le \pi$ at which $|\cos z|$ takes its maximum value.
4. Find the values of z for which

   $(i)\ \exp(\overline{z}) = \overline{\exp(z)}\quad (ii)\ \sinh z + \cosh z = i\quad (iii)\ \cos(i\,\overline{z}) = \overline{\cos i z}$ **(T)** $(iv)\ |\cot z| = 1$
5. Prove that

(T) $(i)\ \sin^{-1} z = -i \log i(z + \sqrt{z^2 - 1})$ $\qquad (ii)\ \cos^{-1} z = -i \log(z + \sqrt{z^2 - 1})$

$(iii)\ \tan^{-1}(z) = \frac{i}{2}\log(\frac{i+z}{i-z}) = \frac{1}{2i}\log(\frac{1+iz}{1-iz})$ $\qquad (iv)\ \cot^{-1}(z) = \frac{i}{2}\log(\frac{z-i}{z+i})$

$(v)\ \sinh^{-1}(z) = \log(z + \sqrt{z^2 + 1})$ $\qquad$ **(T)** $(vi)\ \cosh^{-1}(z) = \log(z + \sqrt{z^2 - 1})$

$(vii)\ \tanh^{-1}(z) = \frac{1}{2}\log(\frac{1+z}{1-z})$ $\qquad (viii)\ \coth^{-1}(z) = \frac{1}{2}\log(\frac{z+1}{z-1})$

6. Test whether the following functions are harmonic and find their harmonic conjugates:

   (T) (i) $u = x^2 - y^2 + x + y - \frac{y}{x^2 + y^2}$ $\qquad$ (ii) u = sin x cosh y + 2 cos x sinh y + x^2 - y^2 + 2xy
7. **(T)** Using that $u(x,y) = 3x^3 + 3x^2 y - 9xy^2 - y^3$ is a homogenous harmonic function, determine an analytic function, as a function of z, whose real part is $u(x,y)$.
8. For each of the following functions find a function f(z) such that f(z) = R $e^{i\varphi}$ is analytic:

   (T) (i) $R = r^2 e^{r\cos\theta}$ $\qquad$ (ii) $\varphi = r^2 \cos\theta \sin\theta$.

P.T.O.

9. **(T)** If f(z) is an analytic function, determine the domain, if any, in which the following functions are harmonic?:
(i) arg f(z) (ii) $|f(z)|$ (iii) ln $|f(z)|$.

10. If the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R (0 < R < ∞), find the radius of convergence of each of the following (k being a fixed natural number):
(T) $(i)\ \sum_{n=0}^{\infty} a_n z^{kn}$ $(ii)\ \sum_{n=0}^{\infty} n^k a_n z^n$ $(iii)\ \sum_{n=0}^{\infty} \frac{a_n}{\underline{|n}} z^n.$

11. **(T)** Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n = \begin{cases} 2^n & if\ \ n\ is\ even \\ 3^n & if\ \ n\ is\ odd. \end{cases}$

12. Find the region of convergence for each of the following power series:
$(i)\sum_{n=0}^{\infty} \frac{z^{2n+1}}{\underline{|n}}$ **(T)** $(ii)\ \sum_{n=0}^{\infty} \frac{\underline{|3n}}{\left(\underline{|n}\right)^3}(z+\pi i)^n$ $(iii)\ \sum_{n=0}^{\infty} (3z-2i)^{3n}$ **(T)** $(iv)\sum_{n=0}^{\infty} \frac{1}{\underline{|n}} z^{n^2}$