Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MSO202A Assignment 3
Introduction To Complex Analysis

The problems marked **(T)** *need an explicit discussion in the tutorial class. Other problems are for enhanced practice.*

1. Evaluate

   (a) $\int_C |z|\bar{z}\, dz,$ where C is the counterclockwise oriented semicircular part of the circle $|z| = 2$ lying in the second and third quadrants.

   (T)(b) $\int_C |z|\frac{z}{\bar{z}}\, dz,$ where C is the clockwise oriented boundary of the part of the annulus $2 \le |z| \le 4$ lying in the third and fourth quadrants.

   (c) $\int_C (z-2a)^n\, dz,$ where C is the semicircle $|z-2a| = R,\ 0 \le \arg(z-2a) \le \pi$.

2. Evaluate the integral $\int_C \frac{1}{\sqrt{z}}\, dz$, in each of the following cases:

   (a) C is the counterclockwise oriented semicircular part of the circle $|z| = 1$ in the upper half plane and $\sqrt{z}$ is defined so that $\sqrt{1} = 1.$

   (T)(b) C is the counterclockwise oriented semicircular part of the circle $|z| = 1$ in the lower half plane and $\sqrt{z}$ is defined so that $\sqrt{1} = -1.$

   (c) C is the clockwise oriented circle $|z| = 1$ and $\sqrt{z}$ is defined so that $\sqrt{1} = 1.$

   (d) C is the counterclockwise oriented circle $|z| = 1$ and $\sqrt{z}$ is defined so that $\sqrt{-1} = i.$

3. Without actually evaluating the integral, prove that

   (T)(a) $\left|\int_C \frac{1}{z^2+1} dz\right| \le \frac{\pi}{3},$ where C is the arc of the circle $|z| = 2$ from z = 2 to z = 2i lying in the first quadrant.

   (b) $\left|\int_C (z^2-1)\, dz\right| \le \pi R(R^2+1),$ where C is the semicircle of radius R > 1 with center at the origin.

4. **(T)**Does Cauchy Theorem hold separately for the real or imaginary part of an analytic function f(z)? Why or why not?

5. About the point z = 0, determine the Taylor series for each of the following functions:
   (T) $(i)\ \sqrt{z+2i}$ $\qquad (ii)\ Log(z^2-3z+2)$

6. About the indicated point z = z_0, determine the Taylor series and its region of convergence for each of the following functions. In each case, does the Taylor series necessarily sums up to the function at every point of its region of convergence?

   $(i)\ \frac{1}{1+z},\ z_0 = 1$ **(T)** $(ii)\ \cosh z,\ z_0 = \pi i$ **(T)** $(iii)\ Log\, z,\ z_0 = -1+i$

7. **(T)**Evaluate the integral $\int_C \frac{dz}{z(z^2+1)}$, for all possible choices of the contour C that does not pass through any of the points z = 0, $\pm\ i$.

8. **(T)** Use Cauchy Theorem for multiply connected domains and Cauchy Integral Formula to evaluate the integral

   $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\ dz$, C: the circle $|z| = 3$ oriented counterclockwise.

9. Evaluate

   (T)(a) $\int_C \frac{e^{2z}}{z(z+1)^4}\ dz$, C: the circle $|z| = 2$ oriented clockwise

   (b) $\int_C \frac{\sin z}{(z+\pi)^{2n}}\ dz$, C: the circle $|z+\pi| = 1$ oriented counterclockwise

10. If u is a harmonic function in $|z| < R\ \ and\ \ 0 < r < R,$ show that

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\ d\theta.$$