

Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MSO202A/MSO202 Assignment 4
Introduction To Complex Analysis

The problems marked **(T)** need an explicit discussion in the tutorial class. Other problems are for enhanced practice.

1. **(T)** Give examples for the following:

- (a) The radius of convergence of Taylor series of a function with center as some point a in the domain of analyticity D of the function is larger than the largest disk $|z - a| < R$ contained in D
- (b) Two Taylor series with different centers represent the same analytic function in the intersection of their disks of convergence.
- (c) The disk of convergence of Taylor series of a function is strictly contained in the domain of analyticity of a function.

2. Evaluate the following integrals on the indicated curves, all of them being assumed to be oriented in the counterclockwise direction:

(T)(a) $\int_C \frac{1}{z^4 - 1} dz$, $C : |z| = 2$ (b) $\int_C \frac{2z^3 + z^2 + 4}{z^4 + 4z^2} dz$, $C : |z - 2| = 4$.

3. Evaluate the following integrals on the square C , oriented in the counterclockwise direction and having sides along the lines $x = \pm 2$ and $y = \pm 2$:

(T)(i) $\int_C \frac{\cos z}{z(z^2 + 8)} dz$ **(T)**(ii) $\int_C \frac{\cosh z}{z^4} dz$.

4. Using Liouville Theorem, show that the functions $\exp(z)$, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ are not bounded in the complex plane \mathbf{C} .

5. Show that every polynomial $P(z)$ of degree n has exactly n zeros in the complex plane.

6. If f is an entire function and $|f(z)| \leq MR^{n_0}$ in $|z| \leq R$, prove that f is a polynomial of degree at most n_0 .

7. Let $f(z)$ be analytic in $|z| \leq R$. Prove that, for $0 < r < R$,

$$f(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr \cos(\theta - \varphi)} f(Re^{i\theta}) d\theta \quad (\text{called Poisson Integral Formula}).$$

8. **(T)** Evaluate $\int_{\Gamma} \frac{1}{z^4} dz$, where Γ is the part of clockwise oriented ellipse $\frac{(x-3)^2}{1} + \frac{y^2}{4} = 1$ lying in the upper half-plane $\{z : \text{Im } z > 0\}$.

9. Find the order of the zero $z = 0$ for the following functions:

(i) $z^2(e^{z^2} - 1)$ **(T)** (ii) $6\sin z^3 + z^3(z^6 - 6)$ **(T)** (iii) $e^{\sin z} - e^{\tan z}$

10. Find the order of all the zeros of the following functions:

(i) $z \sin z$ **(T)** (ii) $(1 - e^z)(z^2 - 4)^3$ **(T)** (iii) $\frac{\sin^3 z}{z}$

11. **(T)** Does there exist a function $f(z)$ (*not identically zero*) that is analytic in $|z| < 1$ and has zeros at the following indicated set of points? Why or why not?

(i) $S_1 = \{\frac{1}{n} : n \text{ is a natural number}\}$ (ii) $S_2 = \{1 - \frac{1}{n} : n \text{ is a natural number}\}$

(iii) $S_3 = \{z : |z| < 1, \operatorname{Re}(z) = 0\}$ (iv) $S_4 = \{z = \frac{1}{2} + iy : -\frac{1}{2} < y < \frac{1}{2}\}$.
