MSO202A: Introduction To Complex Analysis

Instructor:
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Lecture 1

Text Books:

- E. Kreyszig, Advanced Engineering Mathematics, 8^{th} Ed., John Wiley & Sons.
- Ruel V. Churchill, et al: Complex Variables and Applications, McGraw Hill.
- John B. Conway: Functions of One Complex Variable, II Ed., Springer International Student Addition.

Reference Books:

- Jan G. Krzyz: Problems in Complex Variable Theory, American Elsevier Publishing Company.

- Lars V. Ahlfors: Complex Analysis, McGraw Hill.

Supplementary Course Material

- Lecture Notes, Assignments and Course Plan will be available on this course at the webpage http://home.iitk.ac.in/~gp through the link *MSO202A* .

  In the lecture notes, some proofs are marked (*). Such proofs will not be asked in the exams.

Tutorial Classes

- *The assignment problems marked (T) on the assignment sheets will be discussed in the tutorial classes.*
- *The solutions/hints to the assignment problems marked (D) will be made available on the course web-site.*
- *The exercises given in the text books are usually not discussed in the tutorial classes and the students are expected to solve these problems on their own. However, the students can approach the tutor if they have any difficulty in solving such problems.*

Evaluation plan

- There will be 2 pre-announced Quizzes of 40-minutes duration and a weightage of 20% marks for each.
- The End-Course Examination will be of 2-hours duration with a weightage of 60% marks.

Review of Complex Number System

Complex numbers were introduced to have solutions of equations like $x^2+1=0$ which do not possess a solution in the real number system.

A complex number z is an ordered pair (x, y) of real numbers. If $z_1=(x_1, y_1)$, $z_2=(x_2, y_2)$, the elementary operations are defined as

$$z_1+z_2=(x_1+x_2, y_1+y_2)$$
$$z_1=z_2 \text{ } if \text{ } x_1=x_2, y_1=y_2$$
$$-z_1=(-x_1, -y_1)$$
$$z_1z_2=(x_1x_2-y_1y_2, x_1y_2+x_2y_1)$$
$$\overline{z}=(x, -y), \; |z|=\sqrt{x^2+y^2}$$

Notations:

- Throughout in the sequel, denote Complex number$(a,0) \equiv a$, $i \equiv (0,1)$. With these notations, $\mathbf{R} \subseteq \mathbf{C}$, where **R** is set of all real numbers and **C** is set of all complex numbers.

- The Euclidean distance between any two points $z_1, z_2 \in \mathbf{C}$ is defined as $\left|z_1 - z_2\right|$ and is sometimes denoted by $d(z_1, z_2)$.

Note that $\overline{i} = -i, |i| = 1, i^2 = -1$. Thus, the complex number i is the solution of the equation $x^2 + 1 = 0$.

Further, writing $x \equiv (x,0), y \equiv (y,0), i \equiv (0,1)$, it is easily seen by using the definitions of addition and product of complex numbers that $x + i\,y = z$. *This is called the* ***cartesian representation*** *of the complex number z.*

Proposition 1. $z\overline{z} = |z|^2$

Proof: $(x, y).(x, -y) = (x^2 + y^2, 0) \equiv x^2 + y^2 = |z|^2$

Proposition 2. $\frac{1}{z} = (\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2})$, *if* $z \neq 0$

Proof: $\frac{1}{z}\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{|z|^2} = (\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2})$.

Proposition 3. $\operatorname{Re} z = \frac{z + \overline{z}}{2}, \operatorname{Im} z = \frac{z - \overline{z}}{2i}$

Proof. We give here the proof of the second part of the proposition. The first part follows similarly.

$$\frac{\left(z - \overline{z}\right)\overline{i}}{2} = \frac{(0, 2y)(0, -1)}{2} = \frac{(2y, 0)}{2} \equiv y = \operatorname{Im} z.$$

Polar representation of Complex Numbers

With $x = r\cos\theta,\ y = r\sin\theta,\ z = r(\cos\theta + i\sin\theta) = (r\cos\theta, r\sin\theta)$ is called the polar representation of the complex number *z.*

$$r = |z|\ ,$$
$$\theta = angle\ between\ the\ line\ segment\ \ from\ 0\ to\ z\ and$$
$$\quad positive\ real\ axis$$
$$= \arg z$$

It follows immediately that $z_1 = z_2 \Rightarrow r_1 = r_2\ and\ \theta_1 = \theta_2 + 2k\pi$.
Further, if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1),\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, it follows that
$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$.
Thus, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

Similarly, using induction, if

$z_j = r_j(\cos\theta_j + i\sin\theta_j),\ j = 1, 2, \ldots, n,$ then

$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n(\cos(\theta_1 + \ldots + \theta_n) + i\sin(\theta_1 + \ldots + \theta_n))$

Thus, $\arg(z_1 z_2 \ldots z_n) = \arg z_1 + \arg z_2 + \ldots + \arg z_n$.

In particular, $z^n = r^n(\cos n\theta + i\sin n\theta),\ \forall n \geq 0.$

To prove this identity for *n < 0*, we have

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} = \frac{\cos\theta - i\sin\theta}{r} = \frac{1}{r}(\cos\theta + i\sin(-\theta)).$$

So that, for n < 0,

$$z^n = (z^{-1})^{-n} = [\frac{1}{r}(\cos\theta + i\sin(-\theta)]^{-n}$$

$$= r^n[\cos n\theta + i\sin n\theta].$$

Thus, $z^n = r^n[\cos n\theta + i\sin n\theta]$ for all integers n. Taking r = 1 in this identity,

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta\ \ \textit{for all integers}\, n,$

which is called ***De-Moivre's Theorem***.

A special word about argument of a complex number

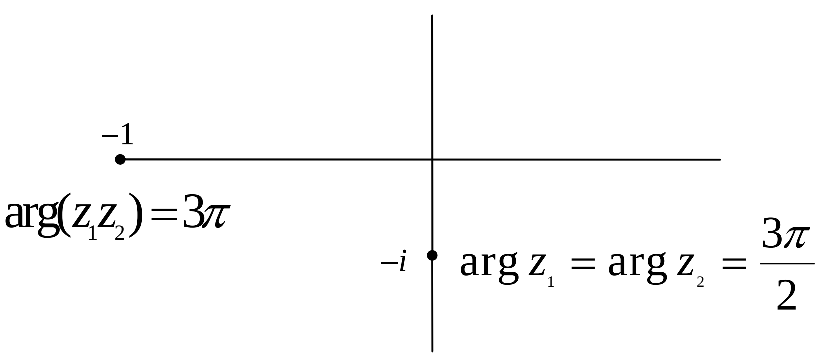
arg z is not a function, since for $z = re^{i\theta}$, $\arg z$ *has all the values* $\theta, \theta \pm 2\pi, \theta \pm 4\pi, \ldots$ so it is not single valued.

The identity

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

has to be interpreted in the sense that for some value of arg on LHS, $\exists$ suitable values of $\arg z_1$ and $\arg z_2$ on RHS so that equality holds. Conversely, for given values of $\arg z_1$ *and* $\arg z_2$ on RHS, $\exists$ suitable values of $\arg(z_1 + z_2)$ on RHS so that equality holds.

For example, if $z_1 = z_2 = -i$ and the values of their arguments are given as $\arg z_1 = \frac{3\pi}{2}$, $\arg z_2 = \frac{3\pi}{2}$, then $z_1 z_2 = -1$ and out of all the values $3\pi \pm 2k\pi,\ k = 0,1,2,\ldots.$ of $\arg(z_1 z_2)$, we must choose $\arg(z_1 z_2) = 3\pi$, so that $\arg z_1 + \arg z_2 = \arg(z_1 z_2)$ holds.



Conversely, if $\arg(z_1 z_2) = 5\pi$ is given, then we can take

$$\arg z_1 = -\frac{\pi}{2} = \frac{3\pi}{2} - 2\pi, \ \arg z_2 = \frac{11\pi}{2} = \frac{3\pi}{2} + 4\pi .$$

To make arg z a function of z in the strict sense of the definition of a function, we restrict the range of arg z as $(-\pi, \pi]$ (or with another convention, some authors restrict this range as $[0, 2\pi)$). Once the range of arg z is so restricted, arg z is denoted by Arg z.

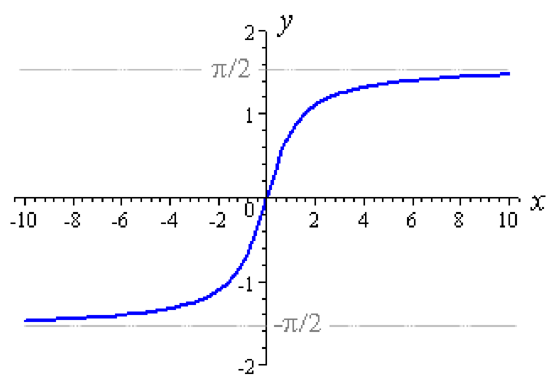
Thus,

$$-\pi < Arg\ z \le \pi$$

(or, $0 \le Arg\ z < 2\pi$ with the other convention).

Remark: If $z = x + iy$, the principal value of $\tan^{-1}\frac{y}{x}$, denoed as $\text{Tan}^{-1}\frac{y}{x}$, satisfies $-\frac{\pi}{2} < Tan^{-1}\frac{y}{x} \le \frac{\pi}{2}$, while $-\pi < Arg\ z \le \pi$. The relation between $Arg\ z$ and $Tan^{-1}\frac{y}{x}$ is therefore given by the following:

$$Arg\,z = \begin{cases} Tan^{-1}\dfrac{y}{x}, & if\ \ x > 0 \\ Tan^{-1}\dfrac{y}{x} + \pi, & if\ \ x < 0,\ y \ge 0 \\ Tan^{-1}\dfrac{y}{x} - \pi, & if\ \ x < 0,\ y < 0 \end{cases}$$



Graph of $Tan^{-1}x$

Remark: Note that, in general,

$$Arg(z_1 z_2) \neq Arg\ z_1 + Arg\ z_2 \qquad (1)$$

For example, with the convention $-\pi < Arg\ z \leq \pi$

if $z_1 = -1,\ z_2 = i,\ then\ Arg\, z_1 = \pi, Arg\, z_2 = \frac{\pi}{2}$, so that

$Arg\, z_1 + Arg\, z_2 = \frac{3\pi}{2}$.

But, $Arg(z_1 z_2) = -\frac{\pi}{2}$. This illustrates (1), when the convention is $-\pi < Arg\ z \leq \pi$.

Similarly, with the convention $0 \leq Arg\ z < 2\pi$, if

$z_1 = -1,\ z_2 = -i,\ then\ Arg\, z_1 = \pi, Arg\, z_2 = \frac{3\pi}{2}$, so that

$Arg\, z_1 + Arg\, z_2 = \frac{5\pi}{2}$.

But, $Arg(z_1 z_2) = \frac{\pi}{2}$. This illustrates (1), when the convention is $0 \leq Arg\ z < 2\pi$.

Solution of the equation $z^n = c$:

Let

$$c = r_0(\cos\theta_0 + i\sin\theta_0)$$

$$and$$

$$z = r(\cos\theta + i\sin\theta)$$

The equation $z^n = c$ gives

$$r^n(\cos n\theta + i\sin n\theta) = r_0(\cos(\theta_0 + 2\pi k) + i\sin(\theta_0 + 2\pi k))$$

$$\Rightarrow r^n = r_0 \ and \ n\theta = \theta_0 + 2k\pi$$

$$\Rightarrow r = r_0^{1/n} \ and \ \theta = \frac{\theta_0 + 2k\pi}{n}, k = 0,1,\ldots,n-1$$

Therefore, the n solutions of the equation $z^n = c$ are

$$z_k = r_0^{1/n}[\cos(\frac{\theta_0 + 2k\pi}{n}) + i\sin(\frac{\theta_0 + 2k\pi}{n})],\ k = 0,1,\ldots,n-1.$$

Since, $|z_k| = r_0^{1/n}$, all the roots of $z^n = c$ lie on the circle $C(0, r_0^{1/n}) \equiv \{z : |z| = r_0^{1/n}\}$. Further, since the angles $\dfrac{\theta_0 + 2k\pi}{n}, k = 0,1,\ldots,n-1$, divide this circle in to n equal sectors, all these roots are equispaced on $C(0, r_0^{1/n})$.

Example: All the roots of $z^n = 1$, called nth roots of unity, can be written as

$$\cos 0 + i\sin 0,\ \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \ldots, \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}$$

or

$$1, \omega_n, w_n^2, \ldots, w_n^{n-1};\ \textit{where}\ w_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

Note that if z_0 is any root of the equation $z^n = c$, then all the roots of this equation are given by

$$z_0, z_0\omega_n, z_0 w_n^2, \ldots, z_0 w_n^{n-1}$$

since, $z_k = |c|^{1/n}[\cos(\frac{\theta_0 + 2k\pi}{n}) + i\sin(\frac{\theta_0 + 2k\pi}{n})],\ k = 0, 1, \ldots, n-1$

gives that

$$z_k\omega_n^l = |c|^{1/n}[\cos(\frac{\theta_0 + 2k\pi + 2l\pi}{n}) + i\sin(\frac{\theta_0 + 2k\pi + 2l\pi}{n})],\quad 0 \le l < n-1$$

whose distinct values are obtained for $k \le k + l < k + n - 1$.

Vector Representation of Complex Numbers

Any complex number $z=(x,y)$ can be represented as the vector

$$z = x\vec{i} + y\,\vec{j} \equiv \vec{r},\ (say).$$

This representation helps in geometrically visualizing addition and subtraction of complex numbers as vectors.

However, it does not help in visualizing the product of complex numbers as this is different from the *vector product* of corresponding vectors.

$$(since,\ if\ \ z_1 = x_1\vec{i} + y_1\vec{j} \equiv \vec{r}_1\ \ and\ \ z_2 = x_2\vec{i} + y_2\vec{j} \equiv \vec{r}_2,\ then$$

$$z_1z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)\ is\ in\ xy-plane\ itself$$

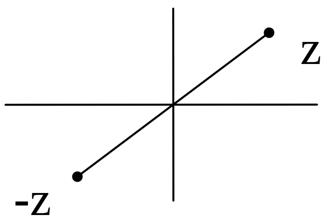
$$while,\ \ for\ the\ corresponding\ vectors\ \vec{r}_1,\ \vec{r}_2,$$

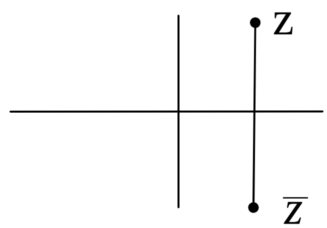
$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} i & j & k \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{vmatrix} = (x_1y_2 - y_1x_2)\vec{k} = (\overline{z}_1 z_2)\vec{k}$$

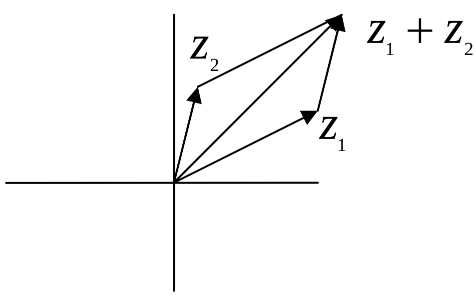
$$is\ \ perpendicular\ to\ xy-plane.)$$

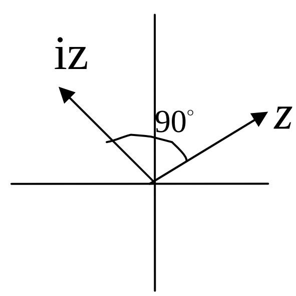
Representation of Points, Curves and Regions by Complex Numbers

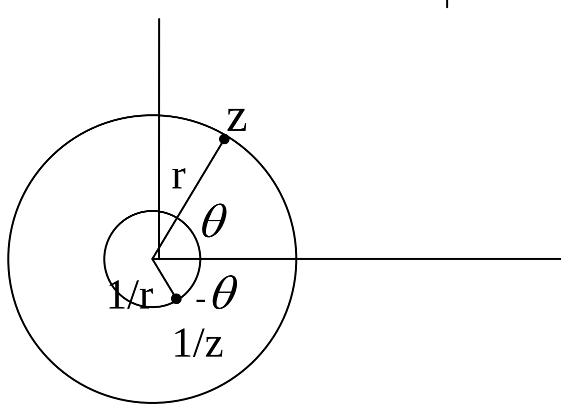
Representation of Points











Equation of a circle and disk in terms of Complex Numbers

Equation of a circle with center z_0 and radius r:
$|z - z_0| = r$
Equation of an open disk with center z_0 and radius r:
$|z - z_0| < r$
Equation of a closed disk with center z_0 and radius r:
$|z - z_0| \le r$

Equation of a Line in terms of Complex Numbers

Equation of a line L passing through $\vec{a}$ and parallel to vector $\vec{b}$ is

$$\vec{r} = \vec{a} + t\vec{b}, \; -\infty < t < \infty$$

or, in terms of notation of a complex variables z, a and b, this equation is

$$z = a + t\,b$$
$$\Rightarrow t = \frac{z-a}{b} \qquad \Rightarrow \operatorname{Im}\left(\frac{z-a}{b}\right) = 0.$$

Thus, equation of the line L is given by

$$L = \left\{ z : \operatorname{Im}\left(\frac{z-a}{b}\right) = 0 \right\}.$$

* Algebraic Structure of Complex Numbers

Field: $(X, +, .)$ is a field if

(i) *(X, +)* is an abelian group.
(ii) *(X – {0}, .)* is an abelian group.
(iii) ‘.’ is distributive over ‘+’.

It is easily verified that *(***C***, +, .)* is a field that contains the field *(***R***, +, .)*.

Ordered Set: (X, <), where, ‘<’ is a relation, is called an ordered set if

(i) One and only one of the statements $x < y$, $x = y$, $y < x$ holds for any x and y.
(ii) ‘<’ is transitive.

Ordered Field: An ordered set X is called an ordered field if

(i) *X* is a field
(ii) *X* is an ordered set
(iii) If $y < z$, then $x + y < x + z$ for all x, y and $z \in X$
(iv) If x > 0, y > 0, then xy > 0.

It is easily verified that (**C**, +, .) is a field as well as an ordered set with respect to dictionary ordering (dictionary order on $\mathcal{C}$ is defined by

$(x_1, y_1) < (x_2, y_2)$ *if either* $x_1 < x_2$ *or if* $x_1 = x_2$ *then* $y_1 < y_2$.

However, (**C**, +, .) is not an ordered field with any order, since in every ordered field 1 is always positive (for, either 1 is positive or -1 is positive and, if -1 is positive, then (-1)(-1) = 1 is positive, which is a contraction), so that -1= (-1, 0) is always negative. Now, either (0, 1) > 0 or - (0, 1) > 0

If $(0,1) > 0$ *then* $(0,1).(0,1) = (-1,0) < (0,0)$,

which implies (**R**, +, .) can not be an ordered field.

If $-(0,1) > 0$ *then* $-(0,1).-(0,1) = (-1,0) < (0,0)$, which implies ($\mathcal{C}$, +, .) can not be an ordered field.

Alternatively, in every ordered field, square of every element is positive. This gives -1 is positive being square of (0,1), a contradiction since -1 is always negative as in the above arguments.