Lecture 10

Maximum Modulus Theorem: *If f is analytic in a domain D and if there is a point* $a \in D$ *such that* $|f(a)| \geq |f(z)|$ *for all* $z \in D$*, then f is a constant function.*

The above theorem can also be stated as 'A non-constant analytic function cannot take its maximum value at any interior point of D***'.***

Corollary 1: If f is analytic on a compact (i.e. closed and bounded) set $K \subset C$***, then*** $|f|$ ***assumes its maximum value on the boundary of K.***

Corollary 2: Let $M(r) = \max_{|z| \leq r} |f(z)|$. *Then,* $M(r) = \max_{|z|=r} |f(z)|$.

Corollary 3: Let $M(r) = \max_{|z|=r} |f(z)|$***. Then, M(r) is an increasing function of r.***

The following proposition is needed for the proof of Maximum Modulus Theorem:

Proposition: *Let* $\varphi(x)$ *be continuous and* $\varphi(x) \le K$ *in* $[a,b]$. *If* $\frac{1}{b-a}\int_a^b \varphi(x)\,dx \ge K$ *(*). Then,* $\varphi(x) \equiv K$ *on* $[a, b]$.

Proof: *Let* $\varphi(c) < K$ *for some* $c \in (a,b)$. Since $\varphi(x)$ is continuous at *c*, for some ε_0,

$$\varphi(x) \le K - \varepsilon_0 \ \textit{ for some interval } (c-\delta_0, c+\delta_0)$$

$$\Rightarrow \int_a^b \varphi(x)\,dx \le 2\delta_0(K-\varepsilon_0) + (b-a-2\delta_0)K.$$

$$= (b-a)K - 2\delta_0\varepsilon_0, \qquad a \textit{ contradiction of } (*).$$

Proof of Maximum Modulus Theorem:

Let $|f(z)| \le |f(a)|$ for all $z \in D$. By Cauchy Integral Formula,

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(w)}{w-a} dw, \;\; \gamma_r(t) = a + re^{it} \subset D, \; 0 \le t \le 2\pi. \qquad (1)$$

Let, $\dfrac{f(w)}{f(a)} = \rho(t) e^{i\varphi(t)}$ on $\gamma_r(t)$. Therefore, by (1),

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \rho e^{i\varphi} \; dt. \qquad (2)$$

Now, $(2) \Rightarrow 1 \le \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \rho \, dt$.

Since, $\rho(t)$ is a continuous function of t and $\rho(t) \le 1$ $(since, \left|f(w)\right| \le \left|f(a)\right|)$. Therefore, by the above proposition, $\rho(t) \equiv 1$ *for all* t.

Taking real part in (2) with $\rho(t) \equiv 1$, $1 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \cos\varphi \; dt$. Since, $\cos\varphi(t)$ is a continuous function of t and $\cos\varphi(t) \le 1$, using the above proposition again, it follows that $\cos\varphi(t) \equiv 1$.

Since $\rho(t) \equiv 1$ *and* $\cos\varphi \equiv 1$ *on* γ_r, $\dfrac{f(w)}{f(a)} = \rho(t) e^{i\varphi(t)}$ on γ_r gives $f(w) = f(a)$ *on* γ_r. This, in view of Isolated Zeros Theorem, gives that $f(w) = f(a)$ everywhere in *D.*

Example. Let $f(z)=e^{e^z}$ and

$D=\{z=x+iy:-\infty<x<\infty,-\pi/2\le y\le\pi/2\}$.

Then, $|f(z)|=e^{\operatorname{Re}e^z}=e^{e^x\cos y}=1,\ if\ \ y=\pm\pi/2\Rightarrow|f(z)|\equiv1$ on boundary of D.

But, $f(x)\to\infty\ as\ x\to\infty$.

Thus, $\max|f(z)|$ need not be assumed on the boundary of D, if D is an unbounded domain.

Minimum Modulus Principle. *If f is analytic in a domain D and $f(z) \neq 0$ for any $z \in D$, then $|f(z)|$ can not assume its minimum value at any point of D, unless $f(z) \equiv constant$.*

Proof: Apply Maximum Modulus Theorem for $g(z) = \dfrac{1}{f(z)}$.

Schwarz Lemma. *Let f be analytic in $|z| \le R$ and satisfies $|f(z)| \le M$ on $|z| = R$. If f(0) = 0, then,*

$$|f(z)| \le \frac{M|z|}{R}, \ \textit{for } |z| < R. \qquad (1)$$

Further, $$|f'(0)| \le \frac{M}{R}. \qquad (2)$$

Equality holds in the above inequalities (1) and (2) for some point in $|z| < R$ iff $f(z) = \frac{M}{R} e^{i\alpha} z$, for some real α.

Proof: Define

$$\varphi(z) = \begin{cases} \dfrac{f(z)}{z} \text{ if } 0 < |z| \le R \\ f'(0) \text{ if } z = 0 \end{cases}$$

Then, $\varphi(z)$ is analytic in $|z| \le R$ (because $\varphi(z)$ is given by the power series $\varphi(z) = f'(0) + \dfrac{f''(0)}{2} z + ...$,which is absolutely convergent at all the points of $|z| \le R$).

$$\Rightarrow |\varphi(z)| \le \frac{M}{R} \text{ for all } z \text{ on } |z| = R$$

$$\Rightarrow |\varphi(z)| \le \frac{M}{R} \text{ for all } z \text{ in } |z| < R, \text{ (by Max. Mod. Theorem)} \quad (3)$$

$$\Rightarrow |f(z)| \le \frac{M|z|}{R} \text{ for all } z \text{ in } 0 < |z| < R$$

The last inequality is trivially true for $z = 0$. This completes the proof of (1).

To prove (2), observe that $|f'(0)| = |\varphi(0)|$,

$$\Rightarrow |f'(0)| \le \frac{M}{R}, \qquad (by\ (3))$$

Equality holds in (1) and (2) *for some point* z_0 *in* $|z| < R$ if and only if $|\varphi(z_0)| = \dfrac{M}{R}$

$\Rightarrow |\varphi(z)|$ assumes its maximum at an interior point z_0 of $|z| < R$.

$\Rightarrow \varphi(z) \equiv \dfrac{M}{R}$ in $|z| < R$ (*by Maximum Modulus Theorem*)

$\Leftrightarrow \varphi(z) = \dfrac{M}{R} e^{i\alpha}$ for some real α in $|z| < R$

$\Leftrightarrow f(z) = \dfrac{Me^{i\alpha}}{R} z$ in $|z| < R$.