Lecture 12

Behaviour of $f(z)$ in the neighbourhood of Pole:

Proposition. *The point a is a pole of order m of f iff* $\lim_{z\to a}(z-a)^m f(z) = A,\ A \neq 0, \infty.$

Proof.

(i) If the point a is a pole of order m of *f,* then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{m} d_n (z-a)^{-n}, d_m \neq 0.$$

$$\Rightarrow \lim_{z\to a}(z-a)^m f(z) = d_m \neq 0, \infty.$$

(ii) If $\lim_{z\to a}(z-a)^m f(z)=A,\ A\neq 0,\infty,$ *then* for every $\varepsilon>0$ there exists a $\delta>0$ such that $|f(z)|<\dfrac{|A|+\varepsilon}{|z-a|^m}$ *in* $0<|z-a|<\delta$.

Therefore,

$$d_n=\frac{1}{2\pi i}\oint_{|w-a|=r}\frac{f(w)}{(w-a)^{-n+1}}dw,\ 0<r<\delta, \text{ gives}$$

$$|d_n|<\frac{1}{2\pi}.\frac{|A|+\varepsilon}{r^{m-n+1}}.2\pi r=\frac{|A|+\varepsilon}{r^{m-n}}\to 0\ as\ r\to 0,\ if\ \ n>m$$

$$\Rightarrow d_n=0,\ if\ \ n>m.$$

For $n = m$, using that $d_k=0\ if\ \ k>m$,

$$A=\lim_{z\to a}(z-a)^m f(z)$$
$$=\lim_{z\to a}(\sum_{n=0}^{\infty}c_n(z-a)^{n+m}+d_1(z-a)^{m-1}+...+d_m)=d_m$$

$\Rightarrow d_m\neq 0\Rightarrow f$ has a pole of order m at the point a.

Proposition. *f has pole of order m at the point a iff 1/f has a zero of order m at a.*

Proof.

(i) Let f have a pole of order m at a. Then, by definition of Pole,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{m} d_n (z-a)^{-n},\ d_m \neq 0$$
$$= (z-a)^{-m}(d_m + d_{m-1}(z-a) + \ldots + d_1(z-a)^m + \sum_{n=0}^{\infty} c_n (z-a)^{n+m})$$
$$= (z-a)^{-m}\varphi(z),$$

where, $\varphi(z)$ is analytic in $|z-a| < R$ for some R and $\varphi(a) = d_m \neq 0$.

Since $\varphi(z)$ is continuous, $\varphi(z) \neq 0$ in some neighbourhood $|z-a| < \delta < R$.

$$\Rightarrow \frac{1}{f(z)} = (z-a)^m \psi(z),\ \textit{where}\ \psi(z)\ \textit{is analytic in}\ |z-a| < \delta$$
$$\&\ \psi(a) \neq 0$$

$\Rightarrow \dfrac{1}{f(z)}$ has a zero of order m at the point a.

(ii) If $\frac{1}{f(z)}$ has a zero of order m at the point a, then

$$\frac{1}{f(z)} = (z-a)^m \varphi(z),$$

where $\varphi(a) \neq 0$ & $\varphi(z)$ *is analytic in* $0 < |z-a| < \delta.$

$$\Rightarrow f(z) = (z-a)^{-m}\psi(z),$$

where $\psi(z)$ *is analytic and nonzero in* $0 < |z-a| < \delta_1.$

(since zeros are isolated)

$$\Rightarrow f(z) = (z-a)^{-m}\left(\sum_{n=0}^{\infty} c_n (z-a)^n\right),\ c_0 \neq 0.$$

$\Rightarrow f(z)$ has a pole of order m at the point a.

Corollary. *f has a pole at the point a iff* $\lim_{z\to a} f(z) = \infty$

Behaviour of $f(z)$ in the neighbourhood of Essential Singularity:

Proposition. *A function f has an essential singularity at a iff $\lim_{z\to a} f(z)$ does not exist.*

Proof.

(i) Let a function f have an essential singularity at a and $\lim_{z\to a} f(z) = A$ *exists.*

If $|A| < \infty$, then f will have a removable singularity at a $\Rightarrow$ a contradiction.

If $A = \infty$, f has a pole at the point a $\Rightarrow$ a contradiction.

Therefore, $\lim_{z\to a} f(z)$ does not exist.

(ii) If $\lim_{z\to a} f(z)$ does not exist, then the point a can not be a pole or removable singularity

$\Rightarrow$ the point a is an essential singularity of f.

Singularity at ∞. *A function f(z) is said to have a singularity (removable, pole or essential) at ∞, if f(1/z) has a singularity (removable, pole or essential respectively) at z = 0.*

Examples.

(i) $P(z) = a_0 + a_1 z + ... + a_n z^n$, $a_n \neq 0$ *has a pole of order n at ∞.*

(ii) e^z *has an essential singularity at ∞.*

(iii) $e^{1/z}$ *is analytic at ∞.*

Nonisolated singularities.

Example: *Let f have a pole at z_n, $n = 1,2,...$, and $z_n \to a$ as $n \to \infty$, then a is a non-isolated singularity of f. Take, e.g., $cosec(\frac{1}{z})$, where $z_n = 1/n\pi$ and a = 0.*

Residues and Integration

Let the function *f(z)* have an isolated singularity at a point 'a' and

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} d_n (z-a)^{-n} \qquad (*)$$

be the Laurent's expansion of *f(z)* in the annulus $0 < |z-a| < R$.

Definition: The ***residue*** of *f(z)* at the point a is defined as

$$\text{Coefficient of } \frac{1}{z-a} = d_1 = \frac{1}{2\pi i} \oint_C f(w)\, dw$$

where, C is any simple, closed, p.w. smooth curve lying in $0 < |z-a| < R$ and enclosing the point a.

Cauchy Residue Theorem.

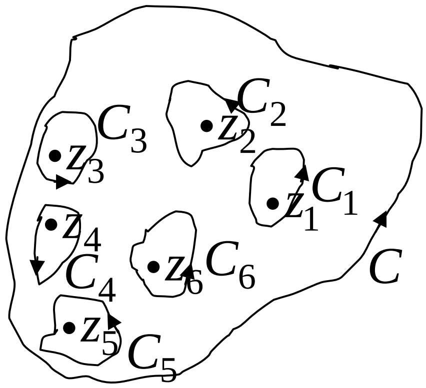
Let a function f(z) be analytic inside and on a simple, closed, p.w. smooth curve C *, except for having finitely singularities at the points* $z_1, z_2, \ldots, z_n$ *enclosed in* C*. Let* p_k *be the residue of f(z) at the point* z_k*. Then,*

$$\oint_C f(w)\,dw = 2\pi i \sum_{k=1}^{n} p_k \,.$$

Proof. By Cauchy Theorem for Multiply Connected Domains,

$$\oint_C f(w)dw = \sum_{k=1}^{n} \oint_{C_k} f(w)dw = 2\pi i \sum_{k=1}^{n} p_k$$

(*the last equality is due to definition of residues*).



Example. Evaluate $\oint\limits_{|z|=2} \frac{5z-2}{z(z-1)}dz$.

Solution.

$$f(z)=\frac{5z-2}{z(z-1)}=-(5-\frac{2}{z})(1-z)^{-1}$$

$$=-(5-\frac{2}{z})(1+z+z^2+...)=\frac{2}{z}-3-3z-....$$

$$\Rightarrow Coeff\ \ of\ \ \frac{1}{z}=\underset{z=0}{res}\ f(z)=2.$$

Further, $f(z)=\frac{5z-2}{z(z-1)}=\frac{5(z-1)+3}{z(z-1)}$

$$=(5+\frac{3}{z-1})(\frac{1}{1+(z-1)})=(5+\frac{3}{z-1})(1+(z-1))^{-1}$$

$$=(5+\frac{3}{z-1})(1-(z-1)+(z-1)^2-...)$$

$\Rightarrow$ Coeff. of $\frac{1}{z-1}$ = 3 $\Rightarrow \underset{z=1}{res}\ f(z)=3$

$\therefore \oint\limits_{|z|=2} \frac{5z-2}{z(z-1)}dz=2\pi i(3+2)=10\pi i$. (*By Cauchy Residue Theroem*)