Lecture 13

Residue at ∞

Definition: *Let C be a simple closed p.w. smooth curve enclosing all finite singularities of f.* Orient C in ***clockwise*** direction. Then, residue of *f* at $z=\infty$ is defined as

$$\operatorname*{res}_{z=\infty} f(z) = \frac{1}{2\pi i}\oint_C f(z)\,dz.$$

Proposition. *The residue of f(z) at* $z=\infty$ *is negative of coefficient of z in the Laurent's expansion of* $f(\frac{1}{z})$ *in* $0<|z|<d$*, where d is so small that* $0<|z|<d$ *does not contain any singularity of* $f(\frac{1}{z})$ *.*

Proof. Let C_R be a circle of sufficiently large radius *R*, enclosing all finite singularities of *f(z)* such that $C_{1/R}$ does not contain any singularity of *f(1/z)* except at *z = 0 and* $\frac{1}{R}<d$ *.* By definition,

$$\underset{z=\infty}{res}\, f(z)=\frac{1}{2\pi i}\oint_{C_R} f(w)\,dw$$

$$\underset{(putting\ w=1/z)}{=} -\frac{1}{2\pi i}\oint_{C_{1/R}} \frac{f(\frac{1}{z})}{z^2}\,dz\ ,$$

($C_{1/R}$ *being counterclockwise oriented circle of radius 1/R*)

$= -$ *coefficient of z in the Laurent's expansion of f(1/z) in* $0<|z|<\frac{1}{R}$.

Example: *Consider the function* $f(z)=\frac{1}{z}+z^n$*.* It has a pole of order 1 at $z=0$ and a pole of order n at $z=\infty$. $\underset{z=0}{Res}\, f(z)=1$ and by the above proposition $\underset{z=\infty}{Res}\, f(z)=-1$.

Proposition. . *If a function f(z) is analytic inside and on a simple, closed, p.w. smooth curve* C, *except for having finitely singularities at the points* $z_1, z_2, \dots, z_n$ *enclosed in C and a singularity at* $z = \infty$. *Then,*

$$\sum_{k=1}^{n} \underset{z=z_k}{res} f(z) + \underset{z=\infty}{res} f(z) = 0 \text{ .}$$

Proof. Follows by using Cauchy Residue Theorem and the definition of residue at ∞.

Techniques for Evaluation of Residues

1. f(z) has a simple pole at z = a. In this case the Laurent's expansion of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \frac{d_1}{z-a}.$$

Therefore,

$$\boxed{\operatorname*{res}_{z=a} f(z) = d_1 = \lim_{z \to a} (z-a) f(z)}$$

If $f(z) = \dfrac{g(z)}{h(z)}$, where *h(z)* has a simple zero at $z = a$ and $g(a) \neq 0$, then *f(z)* has a simple pole at $z = a$. In this case,

$$d_1 = \lim_{z \to a} (z-a) \frac{g(z)}{h(z)} = \lim_{z \to a} \frac{(z-a) g(z)}{h(z) - h(a)} = \frac{g(a)}{h'(a)}$$

If $f(z) = \dfrac{g(z)}{h(z)}$, *where* $h(z)$ *has a simple zero at* $z = a$, $g(a) \neq 0$, *then*

$$d_1 = \frac{g(a)}{h'(a)}$$

Example 1. $f(z)=\cot z$.

The function $f(z)=\cot z=\dfrac{\cos z}{\sin z}$ *has simple poles at the points* $z=n\pi$ *&* $\cos n\pi\neq 0$. *Therefore,*

$$\operatorname*{res}_{z=n\pi} f(z)=\frac{\cos n\pi}{\cos n\pi}=1.$$

(Note the difficulty in finding Laurent's expansion of cot z in deleted neighbourhood of the points $n\pi$)

Example2. $f(z)=\dfrac{z}{z^n-1}$.

Since the function z^n-1 *has simple zeros at the points* $z_k=e^{\frac{2\pi ik}{n}}$, $k=0,\ldots,n-1$, *the function* $f(z)=\dfrac{z}{z^n-1}$ *has simple pole at these points. Therefore,*

$$\operatorname*{res}_{z=z_k} f(z)=\left(\frac{z}{nz^{n-1}}\right)_{z=z_k}=\frac{z_k}{nz_k^{n-1}}=\frac{z_k^2}{n}=\frac{1}{n}e^{\frac{4\pi ik}{n}}.$$

f(z) has a pole of order m at z = a.

Denote $\varphi(z) = (z-a)^m f(z)$. Then, $\varphi(z)$ is analytic in some neighbourhood of the point a. Let, the Taylor series expansion of $\varphi(z)$ in this neighbourhood be

$$\varphi(z) = \varphi(a) + \varphi'(a)(z-a) + \ldots + \frac{\varphi^{(m-1)}(a)}{(m-1)!}(z-a)^{m-1} + \ldots.$$

Therefore, the Laurent's expansion of *f(z)* in the deleted neighbourhood of a is

$$f(z) = \frac{\varphi(z)}{(z-a)^m} = \frac{\varphi^{(m-1)}(a)}{(m-1)!}\frac{1}{(z-a)} + \ldots + \varphi(a)\frac{1}{(z-a)^m}$$
$$+ \textit{nonnegative powers of } (z-a)$$

$$\Rightarrow d_1 = \textit{coefficient of } \frac{1}{z-a} = \frac{\varphi^{(m-1)}(a)}{(m-1)!}$$
$$= \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-a)^m f(z)]_{z=a}$$

Example 1. *Find the residue of the function* $f(z)=\dfrac{z^3+5}{z(z-1)^3}$ *at* $z=1$.

Solution: The function *f(z)* has a pole of order 3 at *z* = 1. Therefore,

$$\underset{z=1}{res}\, f(z)=\frac{1}{2!}\left[\frac{d^2}{dz^2}\left(\frac{z^3+5}{z}\right)\right]_{z=1}$$

$$=\frac{1}{2!}\left[\frac{d}{dz}\left(\frac{3z^2.z-(z^3+5)}{z^2}\right)\right]_{z=1}$$

$$=\frac{1}{2!}\left[\frac{d}{dz}\left(\frac{2z^3-5)}{z^2}\right)\right]_{z=1}$$

$$=\frac{1}{2!}\left[\frac{6z.z^2-2z(2z^3-5)}{z^4}\right]_{z=1}$$

$$=\frac{1}{2!}\left[\frac{2z^3+10}{z^4}\right]_{z=1}=\frac{1}{2!}\times 12=6$$

Example 2. *Find the residue of the function* $f(z)=\dfrac{1}{(1+z^2)^n}$ *at all its poles*.

Solution. *f(z)* has poles of order n at *z* = i and *z* = - i . Therefore,

$$\operatorname*{res}_{z=i} f(z)=\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-i)^n\frac{1}{\left(z^2+1\right)^n}\right]_{z=i}$$

$$=\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[\frac{1}{(z+i)^n}\right]_{z=i}$$

$$=\frac{1}{(n-1)!}\left[(-1)^{n-1}\frac{n(n+1)\ldots(2n-2)}{(z+i)^{2n-1}}\right]_{z=i}$$

$$=\frac{(-1)^{n-1}}{(n-1)!}\frac{(2n-2)!}{(n-1)!}\frac{1}{(2i)^{2n-1}}=\frac{(2n-2)!\times(-1)^{n-1}}{[(n-1)!]^2\,2^{2n-1}}\frac{1}{i}\frac{1}{(i)^{2n-2}}$$

$$=-i\frac{(2n-2)}{[(n-1)!]^2\,2^{2n-1}}$$

Evaluation of Real Integrals by Residue Method

(I) Integrals of the form $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$***; R is rational function of its arguments:***

Let $z = e^{i\theta}$. Then, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$.

Therefore,

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \frac{1}{i} \int_{|z|=1} \frac{R(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z}))}{z}\, dz$$

$$= \frac{1}{i} \int_{|z|=1} R^*(z)\, dz \qquad (*)$$

where $R^*(z)$ is a rational function of z, analytic in $|z| < 1$, except for having finitely many poles N, at the points $z_k, k = 1, 2, \ldots, N$, where $N \leq$ degree of the denominator of $R^*(z)$.

By (*),

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = 2\pi \sum_{k=1}^{N} \underset{z=z_k}{res}\, R^*(z).$$

It therefore follows that If α_k is the order of pole at z_k, then

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = 2\pi \sum_{k=1}^{N} \frac{1}{(\alpha_k - 1)!} \lim_{z \to z_k} \frac{d^{\alpha_k - 1}}{dz^{\alpha_k - 1}} \left[(z - z_k)^{\alpha_k} R^*(z) \right].$$

Example. *Evaluate* $I = \int_0^{\pi} \frac{a}{a^2 + \sin^2\varphi}\, d\varphi$, *$a > 0$.*

Solution.

$$I = \int_0^{\pi} \frac{2a}{1 + 2a^2 - \cos 2\varphi} d\varphi = \int_0^{2\pi} \frac{a}{1 + 2a^2 - \cos\theta} d\theta$$

.

$$= \int_{|z|=1} \frac{2ai}{z^2 - 2z(1 + 2a^2) + 1} dz$$

The integrand has simple poles at the points $1 + 2a^2 \pm 2a\sqrt{1 + a^2}$. *The positive sign corresponds to a point outside* $|z| = 1$*. The pole* $A = 1 + 2a^2 - 2a\sqrt{1 + a^2}$ *lies in* $|z| < 1$*, and*

$$\operatorname*{res}_{z=A} \frac{2ai}{z^2 - 2z(1 + 2a^2) + 1}$$

$$= \lim_{z \to A} \frac{2ai(z - A)}{z^2 - 2z(1 + 2a^2) + 1} = \lim_{z \to A} \frac{2ai}{(z - 1 - 2a^2 - 2a\sqrt{1 + a^2})}.$$

$$= \frac{1}{2i\sqrt{a^2 + 1}}$$

Therefore, $I = 2\pi i \times \frac{1}{2i\sqrt{a^2 + 1}} = \frac{\pi}{\sqrt{a^2 + 1}}$.