Lecture 14

(II) Integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$**.**

The integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined as

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_{-a}^{c} f(x)\,dx + \lim_{b\to\infty}\int_{c}^{b} f(x)\,dx.$$

If the limit on RHS does not exist, or gives an indeterminate form $\infty-\infty$, $\int_{-\infty}^{\infty} f(x)\,dx$ does not exist. In this case, we define Cauchy Principle Value of $\int_{-\infty}^{\infty} f(x)\,dx$ as

$$p.v.\int_{-\infty}^{\infty} f(x)\,dx = \lim_{r\to\infty}\int_{-r}^{r} f(x)\,dx.$$

Example. *For f(x) = x, the integral* $\int_{-\infty}^{\infty} f(x)\,dx$ *does not exist but*

$$p.v.\int_{-\infty}^{\infty} x\,dx = \lim_{r\to\infty}\int_{-r}^{r} x\,dx = \lim_{r\to\infty}\left(\frac{r^2}{2}-\frac{r^2}{2}\right) = 0.$$

Note that if $\int_{-\infty}^{\infty} f(x)\,dx$ *exists,* $\int_{-\infty}^{\infty} f(x)\,dx = p.v.\int_{-\infty}^{\infty} f(x)\,dx.$

Using the method of residues, the Principle Value of above type of real integrals can be found. We need the following Proposition for this purpose:

Proposition. *Let*

(i) f(z) be analytic in Im z > 0, except for having finitely many singularities in Im z > 0

(ii) $\left| f(z) \right| < \dfrac{M}{\left| z \right|^{1+\delta}}$, *for* $\left| z \right| > R_0$, *for some* $M, R_0, \delta > 0.$

Then, $\lim\limits_{R\to\infty} \int\limits_{C_R} f(w)\, dw = 0,\quad$ *where* $C_R : \left| w \right| = R,\ \mathrm{Im}\, w > 0.$

Remarks.

(i) *The conditions of the proposition are satisfied if*

(a) f(z) is analytic in some neighbourhood of $z = \infty$ *(i.e. outside of some disk centered at origin) and, at* $z = \infty$*, f(z) has a zero of order* ≥ 2*.*

For, in this case, Laurent's expansion of f(z) in the neighbourhood of z = ∞, is of the form

$$f(z) = \frac{d_2}{z^2} + \frac{d_3}{z^3} + \ldots \equiv \frac{\psi(z)}{z^2},\ \textit{where}\ \left| \psi(z) \right| < M \quad \textit{for}\ \left| z \right| > R_0$$

$\Rightarrow$the conditions of the proposition $\left| f(z) \right| < \dfrac{M}{\left| z \right|^2}$ *for* $\left| z \right| > R_0$ is satisfied if *f(z)* has a zero of order ≥ 2 at $z = \infty$.

(b) $f(z) = \dfrac{P(z)}{Q(z)}$, $P(z), Q(z)$ *polynomials, and*

degree of denominator – degree of numerator ≥ 2.

In this case, $f(z)$ *has a zero of order* ≥ 2 *at* $z = \infty$, so that by (i), the conditions of the proposition are satisfied

Proof of the Proposition. For $R > R_0$,

$$\left| \int_{C_R} f(w)\,dw \right| \leq \int_{C_R} |f(w)||dw| < \frac{M}{R^{1+\delta}}.\pi R = \frac{\pi M}{R^{\delta}} \to 0 \ as \ R \to \infty.$$

Theorem. *Let*

(i) f(z) be analytic in $\operatorname{Im} z \geq 0$ *except for having finitely many singular points* z_k, $k = 1, 2, \ldots, N$ *in* $\operatorname{Im} z > 0$

(ii) $|f(z)| < \dfrac{M}{|z|^{1+\delta}}$ *for* $|z| > R_0$, *for some* $R_0, M, \delta > 0$

Then, $p.v. \int_{-\infty}^{\infty} f(x)\, dx$ *exists and*

$$p.v. \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{res} \left[f(z) \right].$$

Proof. Let $|z_k| < R_0$ *for* $k = 1, \ldots, N$. For $R > R_0$, let

$$\Gamma_R : \{ z = x + iy : -R \leq x \leq R, y = 0 \} \cup \{ z : |z| = R, \operatorname{Im} z > 0 \}$$

By Cauchy Residue Theorem,

$$\int_{\Gamma_R} f(z) dz = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z) dz = 2\pi i \sum_{k=1}^{N} \underset{z=z_k}{res} \left[f(z) \right]$$

where, C_R is the counterclockwise oriented semicircle $\{ z : |z| = R, \operatorname{Im} z > 0 \}$.

Using the proposition, it follows that the limit of second integral on LHS is 0 as $R \to \infty$.

$$\therefore p.v. \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{k=1}^{N} \underset{z=z_k}{res} \left[f(z) \right]$$

Example. *Evaluate* $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$

Solution. Since the above integral exists, $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx = p.v. \int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$. Let f(z) = $\frac{1}{z^4+1}$. It has singular points at $z_k = (-1)^{1/4} = e^{\frac{2\pi ik+\pi i}{4}}$, $k = 0,1,2,3$. Therefore,

$$z_0 = e^{\pi i/4}, z_1 = e^{3\pi i/4}, z_2 = e^{5\pi i/4} = e^{-3\pi i/4}, z_3 = e^{7\pi i/4} = e^{-\frac{\pi i}{4}}.$$

Only z_0 *and* z_1 lie in $\operatorname{Im} z > 0$ and the conditions of the previous theorem are satisfied.

$$\therefore I = 2\pi i \left[\underset{z=e^{\pi i/4}}{res} \frac{1}{1+z^4} + \underset{z=e^{3\pi i/4}}{res} \frac{1}{1+z^4} \right]$$

$$= 2\pi i \left[\left(\frac{1}{4z^3} \right)_{e^{\pi i/4}} + \left(\frac{1}{4z^3} \right)_{e^{3\pi i/4}} \right]$$

$$= \frac{2\pi i}{4} \left[\frac{1}{e^{3\pi i/4}} + \frac{1}{e^{9\pi i/4}} \right] = \frac{\pi i}{2} \left[-e^{\pi i/4} + e^{-2\pi i} . e^{-\pi i/4} \right]$$

$$= \frac{\pi i}{2} \left[-e^{\pi i/4} + e^{-\pi i/4} \right] = \frac{2\pi i}{4} \left(-2i \sin \frac{\pi}{4} \right) = \frac{\pi}{\sqrt{2}}.$$

Note. *If f(x) is an even function, then* $\int_0^{\infty} f(x)\, dx$ *can also be evaluated by this method.*

(III) Integrals of the form $\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,dx$***,*** $\alpha > 0$

(Fourier Integrals)

We need the following result:

Jordan's Lemma: Let,

(i) $f(z)$ ***be analytic in*** $\operatorname{Im} z > 0$ ***except for having finitely many singular poits***

(ii) $f(z) \to 0$ *uniformly as* $z \to \infty$ *in* $\{z : 0 < \arg z < \pi\}$***.***

Then, for $\alpha > 0$***,*** $\lim_{R\to\infty} \int_{C_R} e^{i\alpha w} f(w)\,dw = 0$***, where*** C_R ***is the semicircle*** $|z| = R$, $\operatorname{Im} z > 0$***.***

Proof. We use the Jordan's inequality

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1, \quad \textit{for } 0 \le \theta \le \pi/2$$

(Proof of Jordan's inequality: we first show that if f(t) is ↓ as t↑, then $F(t) = \frac{1}{t}\int_0^t f(x)\,dx \quad t > 0$, is also decreasing with t↑. Obviously, $F(t) > f(t)$ *for all* t. Therefore,

$$F'(t) = -\frac{1}{t^2}\int_0^t f(x)\,dx + \frac{f(t)}{t} = -\frac{F(t)}{t} + \frac{f(t)}{t} < 0 \Rightarrow F(t)\downarrow as\ t\uparrow.$$

Applying this result to $\cos\theta\ in\ 0 \le \theta \le \pi/2$ (since $\cos\theta$ is $\downarrow$ in this interval), it follows that

$$\frac{1}{\theta}\int_0^\theta \cos x\,dx = \frac{\sin\theta}{\theta}\ is \downarrow in\ 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1)$$

Now, by hypothesis,

$$|f(z)| < \mu(R)\ on\ C_R,\ where\ \mu(R) \to 0\ as\ R \to \infty.$$

$$\left|\int_{C_R} e^{i\alpha w} f(w)\,dw\right| < R\mu_R \int_0^\pi \left|e^{i\alpha w}\right| d\varphi = R\mu_R \int_0^\pi e^{-\alpha R\sin\varphi}\,d\varphi$$

$$\underset{(using\ f(\varphi)=f(\pi-\varphi))}{=} 2R\mu_R \int_0^{\pi/2} e^{-\alpha R\sin\varphi}\,d\varphi$$

$$\underset{(using\ Jordan's\ inequality)}{\Rightarrow} \left|\int_{C_R} e^{i\alpha w} f(w)\,dw\right| \le 2R\mu_R \int_0^{\pi/2} e^{-\alpha R.\frac{2\varphi}{\pi}}\,d\varphi .$$

$$= \frac{\pi}{\alpha}\mu_R(1 - e^{-\alpha R}) \to 0\ as\ R \to \infty$$

Theorem. *Let f(z) be analytic in* $\operatorname{Im} z \geq 0$ *except for having finitely many singularities in* $\operatorname{Im} z > 0$. *Let f(z) satisfy the conditions of Jordan's Lemma. Then, the integral* $p.v. \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, dx,\ \alpha > 0,$ *exists and is given by*

$$p.v. \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, dx = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{res} \left[e^{i\alpha z} f(z) \right]$$

where z_k *are the singularities of f(z) in the upper half plane.*

Proof. Let R_0 be such that $|z_k| < R_0$ for all k = 1,2,…, N. By Cauchy Residue Theorem,

$$\int_{-R}^{R} e^{i\alpha x} f(x)\, dx + \int_{C_R} e^{i\alpha w} f(w)\, dw = 2\pi i \sum_{k=1}^{n} \underset{z=z_k}{res} \left[e^{i\alpha z} f(z) \right].$$

Taking limit $R \to \infty$ and using Jordan's Lemma, the Theorem follows.

Example 1. *Evaluate* $I = \int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2 + a^2}\, dx\ ;\ \alpha > 0,\ a > 0.$

Solution. $I = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2 + a^2}\, dx = \operatorname{Re} I_1\ (say).$

The function $f(z) = \frac{1}{z^2 + a^2} \to 0$ *as* $z \to \infty$ *in the upper half plane and it has a pole of order* 1 *at* $z = ia$ *in the upper half plane.*

$$\therefore I_1 = 2\pi i \operatorname*{res}_{z=ia}\left[e^{i\alpha z} f(z) \right] = 2\pi i \operatorname*{res}_{z=ia}\left[e^{i\alpha z} \frac{1}{z^2 + a^2} \right]$$

$$= 2\pi i. \frac{e^{-\alpha a}}{2ia} = \frac{\pi}{a} e^{-\alpha a}$$

$$\Rightarrow I = \operatorname{Re} I_1 = \frac{\pi}{a} e^{-\alpha a}.$$

Example 2. *Evaluate* $I = \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{\left(x^2+a^2\right)(x-ia)}\, dx \;;\; \alpha > 0,\, a > 0.$

Solution. *The function* $f(z) = \frac{1}{\left(z^2+a^2\right)(z-ia)} \to 0$ *as* $z \to \infty$ *in the upper half plane and it has a pole of order* 2 *at* $z = ia$ *in the upper half plane.*

$$\therefore I = 2\pi i \operatorname*{res}_{z=ia}\left[\frac{e^{i\alpha z}}{\left(z^2+a^2\right)(z-ia)}\right] = 2\pi i \left\{\frac{d}{dz}\left(\frac{e^{i\alpha z}}{z+ia}\right)\right\}_{z=ia}$$

$$= \left[\frac{i\alpha e^{i\alpha z}(z+ia) - e^{i\alpha z}}{\left(z+ia\right)^2}\right]_{z=ia} = e^{-a^2}\left[\frac{-2a^2-1}{\left(2ia\right)^2}\right] = e^{-a^2}\,\frac{1+2a^2}{4a^2}.$$

(Note that the point z = -ia is in the lower half plane, so residue at this point need not be computed for the evaluation of the integral)

Remarks.

(i) *If f(x) is even,*

$$\int_0^{\infty} f(x)\cos\alpha x\, dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx = \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\, dx$$

$$= \frac{1}{2}\operatorname{Re}\left[2\pi i\sum_{k=1}^{N} \underset{z=z_k}{res}\left[f(z)e^{i\alpha z}\right]\right] = -\pi\operatorname{Im}\sum_{k=1}^{N} \underset{z=z_k}{res}\left[f(z)e^{i\alpha z}\right].$$

(ii) *If f(x) is odd,*

$$\int_0^{\infty} f(x)\sin\alpha x\, dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx = \frac{1}{2}\operatorname{Im}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\, dx$$

$$= \frac{1}{2}\operatorname{Im}\left[2\pi i\sum_{k=1}^{N} \underset{z=z_k}{res}\left[f(z)e^{i\alpha z}\right]\right] = \pi\operatorname{Re}\sum_{k=1}^{N} \underset{z=z_k}{res}\left[f(z)e^{i\alpha z}\right].$$

(IV) Fourier Integrals having Singularities at Real Axis

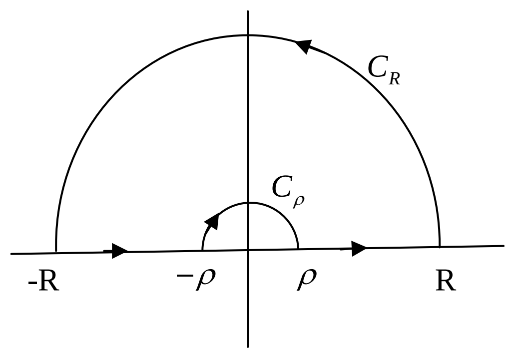
We illustrate this case by considering the evaluation of the integral $I = \int_0^\infty \frac{\sin \alpha x}{x}\, dx,\ \alpha \neq 0.$

Note that $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin \alpha x}{x}\, dx = \frac{1}{2}\operatorname{Im}\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\, dx$.

Let contour of integration be as shown in the figure and

$\Gamma_{\rho,R} = [-R,-\rho] \cup C_\rho^- \cup [\rho, R] \cup C_R^+$ *, where* $R > \rho$

Then, by Cauchy Theorem, $\int_{\Gamma_{\rho,R}} \frac{e^{i\alpha z}}{z} dz = 0$



$$\Rightarrow \int_{[-R,\ \rho]} \frac{e^{i\alpha x}}{x}\, dx + \int_{[\rho,\ R]} \frac{e^{i\alpha x}}{x}\, dx + \int_{C_\rho^-} \frac{e^{i\alpha w}}{w} dw + \int_{C_R} \frac{e^{i\alpha w}}{w} dw = 0. \ (*)$$

The last integral tends to 0 as $R \to \infty$ *(by Jordan's Lemma).*

Further, $\displaystyle\int_{C_\rho^-} \frac{e^{i\alpha w}}{w} dw \underset{(putting\ w=\rho e^{i\varphi})}{=} i\int_{\pi}^{0} e^{i\alpha\rho(\cos\varphi + i\sin\varphi)}\, d\varphi.$

Since the integrand is continuous function of ρ *in the interval* $[0, \pi]$, *the above identity gives*

$$\lim_{\rho\to\infty} \int_{C_\rho^-} \frac{e^{i\alpha w}}{w} dw = i\int_{\pi}^{0} d\varphi = -\pi i.$$

Therefore, by (*), $\displaystyle p.v.\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\, dx = \pi i \Rightarrow \int_{0}^{\infty} \frac{\sin\alpha x}{x}\, dx = \frac{\pi}{2}$.