Lecture 15

Counting Number of Zeros and Poles of a Function in a Given Simple Closed Curve.

Proposition. *Let f be analytic inside and on a simple, closed, p.w. smooth and counterclockwise oriented curve C, except possibly for having finitely many poles inside C. Let f have no zeros on C. Then,*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N_0 - N_\infty$$

where,

$N_0 =$ *Number of zeros of f inside C (Counted according to their multiplicity)*
$N_\infty =$ *Number of poles of f inside C (Counted according to their multiplicity).*

Proof. Let the point a be zero of order n of the function f. Then,

$$f(z)=(z-a)^n f_1(z),\ \ f_1(z)\neq 0 \ in\ a\ neighbourhood\ of\ a$$

$$\Rightarrow f'(z)=n(z-a)^{n-a}+(z-a)^n f_1'(z)$$

$$\Rightarrow \frac{f'(z)}{f(z)}=\frac{n}{z-a}+\frac{f_1'(z)}{f_1(z)}=\frac{n}{z-a}+\sum_{n=0}^{\infty}c_n(z-a)^n,$$ in a deleted neighbourhood of a.

$$\Rightarrow a\ is\ a\ pole\ of\ \ f\ \ and\ \underset{z=a}{res}\frac{f'(z)}{f(z)}=n.$$

Similarly, let the point b be a pole of order m of the function *f.* Then,

$$f(z)=\frac{b_1}{(z-b)^m}+\ldots+\frac{b_m}{(z-b)}+b_{m+1}+b_{m+2}(z-b)+\ldots,,where\ b_1\neq 0$$

$$=\frac{f_2(z)}{(z-b)^m}$$

where, $f_2(z)$ *is analytic at b and* $f_2(b)=b_1\neq 0 \quad \Rightarrow f_2(z)\neq 0$ in some neighbourhood of b .

$$\Rightarrow \frac{f'(z)}{f(z)}=-\frac{m}{z-b}+\frac{f_2'(z)}{f_2(z)}=-\frac{m}{z-b}+\sum_{n=0}^{\infty}c_n^*(z-b)^n,$$ in a deleted neighbourhood of b

$$\Rightarrow b\ is\ a\ pole\ of\ \ f\ \ and\ \underset{z=b}{res}\frac{f'(z)}{f(z)}=-m.$$

Thus, the singularities of $\frac{f'(z)}{f(z)}$ inside C are zeros and poles of *f, with residues at them respectively equal to their multiplicities or negative of their multiplicities.*

Now, let $a_1,\ldots,a_{m_k}$ be the zeros of f with multiplicities $m_1,\ldots,m_k$, $m_1+\ldots+m_k=N_0$ and $b_1,\ldots,b_{n_l}$ be the poles of f with multiplicities $n_1,\ldots,n_l$, $n_1+\ldots+n_l=N_\infty$. Using the above arguments, it follows that sum of residues of $\frac{f'(z)}{f(z)}$ at zeros and poles of f is N_0-N_∞.

Therefore, by Cauchy Residue Theorem,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N_0 - N_\infty .$$

Argument Principle.

Let f be analytic inside and on a simple, closed, p.w. smooth counterclockwise oriented curve C , except for having finitely many poles inside C. Let f have no zeros on C. Then,

$$\frac{1}{2\pi}\Delta_C[\arg f(z)] = N_0 - N_\infty$$

where,

$N_0 =$ *Number of zeros of f inside C (Counted according to their multiplicity)*

$N_\infty =$ *Number of poles of f inside C (Counted according to their multiplicity)*

and

$\Delta_C[\arg f(z)]$= Change in $\arg f(z)$ as z describes C once in counterclockwise direction

Remark: $\frac{1}{2\pi}\Delta_C[\arg f(z)]$ is **winding number of the image curve** $\Gamma = f(C)$ **around origin**. Therefore, Argument Principle relates winding number of the image curve $\Gamma = f(C)$ around origin with the number of zeros and poles of the function $f(z)$.

Proof. Let $z = z(t)$, $a \le t \le b$ be the parametric representation of C. Let $\Gamma = f(C)$. Then, a parametric representation of Γ is $w(t) = f(z(t))$, $a \le t \le b$.

Since, $w'(t) = f'(z(t))\, z'(t)$, $a \le t \le b$,

$$\oint_C \frac{f'(z)}{f(z)} dz = \int_a^b \frac{f'(z(t))}{f(z(t))} z'(t)\, dt = \int_a^b \frac{w'(t)}{w(t)} dt = \oint_\Gamma \frac{dw}{w} \quad (1)$$

Now, consider another parametric representation of Γ given by

$$w(\tau) = \rho(\tau) e^{i\varphi(\tau)}, \; c \le \tau \le d.$$

Obviously,

$\varphi(d) - \varphi(c) =$ Change in the argument of w as it describes Γ once in anticlockwise direction

= Change in the argument of $w = f(z)$ as z describes C once

= $\Delta_C[\arg f(z)]$.

Since $w'(\tau) = \rho'(\tau) e^{i\varphi(\tau)} + i\rho(\tau) e^{i\varphi(\tau)} \varphi'(\tau)$

$\Rightarrow \dfrac{w'(\tau)}{w(\tau)} = \dfrac{\rho'(\tau)}{\rho(\tau)} + i\, \varphi'(\tau)$ it follows that

$$\oint_\Gamma \frac{dw}{w} = \oint_\Gamma \left[\frac{\rho'(\tau)}{\rho(\tau)} + i\, \varphi'(\tau)\right] d\tau = [\log \rho(\tau)]_c^d + i[\varphi(\tau)]_c^d$$

$$= i[\varphi(d) - \varphi(c)] = i\Delta_C[\arg f(z)]. \quad (2)$$

(1) and (2) imply

$$\oint_C \frac{f'(z)}{f(z)} dz == i\Delta_C[\arg f(z)] \qquad (*)$$

Since, by previous proposition,

$$\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i(N_0 - N_\infty),$$

the desired equation $\frac{1}{2\pi}\Delta_C[\arg f(z)] = N_0 - N_\infty$ of Argument Principle follows by (*).

Geometrical Interpretation of Argument Principle

Since,

$$\Delta_C[\arg f(z)] = \varphi(d) - \varphi(c)$$
$$= 2\pi \times \textit{winding number of } \Gamma \textit{ around origin}$$

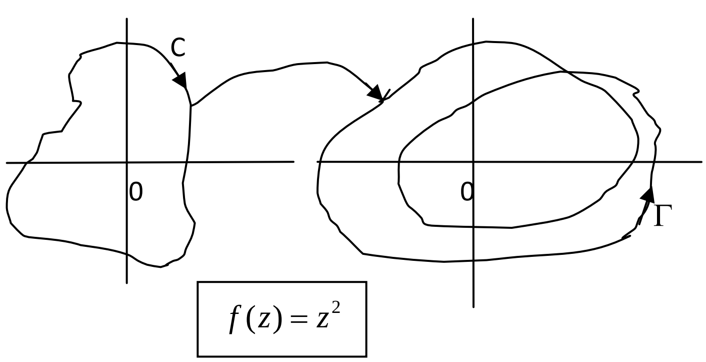
$\Rightarrow$ Winding number of the curve $\Gamma = f(C)$ around origin $= N_0 - N_\infty$.

Notes.

(1) *Let f be analytic inside and on C. Then,*

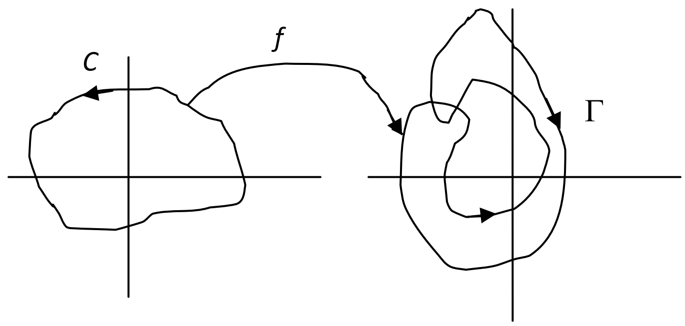
(a) *f has m zeros inside C iff $\Gamma = f(C)$ winds m times around origin*

Example. Let *$f(z) = z^2$*. Then, *$f(z)$* has 2 zeros in a simple closed curve C enclosing origin. It follows that $\Gamma = f(C)$ winds 2-times around the origin.



(b) *f has no zeros inside C iff $\Gamma = f(C)$ does not wind around origin*

Example. Let *f(z)* have no zeros in a simple closed curve C enclosing origin. It follows that $\Gamma = f(C)$ does not wind around origin.



(2) *Let f be nonzero inside C. Then,*

(a) *f has no poles inside C iff* $\Gamma = f(C)$ does not wind around origin

(b) *f has n poles inside C iff* $\Gamma = f(C)$ winds n times around origin.

Example. $f(z) = \dfrac{1}{z^2}, \; C : |z| = 1$

Rouche's Theorem

Let f and g be analytic inside and on a simple, closed, p.w. smooth curve C. If $|f(z)| > |g(z)|$ for all points z on C, then f(z) and f(z) + g(z) have same number of zeros inside C. It is assumed that C is oriented in anticlockwise direction.

Proof. Observe that

(i) f has no zeros on C $(\because \textit{ for } z \in C, |f(z)| > |g(z)| \geq 0)$

(ii) $f + g$ has no zeros on C
$(\because \textit{ for } z \in C, |f(z) + g(z)| \geq |f(z)| - |g(z)| > 0)$.

Let $N_f = \textit{Number of zeros of } f \textit{ inside } C$
$N_{f+g} = \textit{Number of zeros of } f + g \textit{ inside } C$.

By Argument Principle, since f and $f + g$ have no poles inside C,

$$(1) \quad \begin{cases} \dfrac{1}{2\pi}\Delta_C[\arg f(z)] = N_f \\ \dfrac{1}{2\pi}\Delta_C[\arg(f(z) + g(z))] = N_{f+g} \end{cases}$$

Now,

$$(2) \qquad \Delta_C \arg[f(z)+g(z)] = \Delta_C \arg[(1+\frac{g(z)}{f(z)})f(z)]$$

$$= \Delta_C \arg[(1+\frac{g(z)}{f(z)})] + \Delta_C \arg[f(z)]$$

Let $w = 1+\frac{g(z)}{f(z)}$ maps C on to Γ. Then, Γ lies inside the circle $|w-1| = 1$ (since, for any point $w \in \Gamma$, $|w-1| = \left|1+\frac{g(z)}{f(z)}-1\right| < 1$) .

$\Rightarrow \Gamma$ does not wind around origin

$\Rightarrow \Delta_C \arg[1+\frac{g(z)}{f(z)}] = 0.$

$\underset{(by\ (1)\ and\ (2))}{\Rightarrow} N_f = N_{f+g}$

Thus, f and $f+g$ have the same number of zeros inside C.

Example. *Find the number of zeros of* $P(z) = z^{10} - 6z^7 + 3z^3 + 1$ *in* $|z| < 1$.

Solution. Let $f(z) = -6z^7$ *and* $g(z) = z^{10} + 3z^3 + 1$. Then, on $|z| = 1$,

$$|g(z)| < 5 \quad and \quad |f(z)| = 6$$

Therefore, the conditions of Rouche's Theorem are satisfied. Since *f(z)* has 7 zeros in $|z| < 1$, *f(z) + g(z) = P(z)* also have 7 zeros in $|z| < 1$.

Exercise. *Find the number of zeros of* $P(z) = z^7 + 4z^6 - 15z^5 + 7z^3 + 2$ in $1 < |z| < 2$.

Hint.

$$On\ |z| = 1,\ |z^7 + 4z^6 + 7z^3 + 2| < 14 < |-15z^5| = 15$$

$$\Rightarrow P\ has\ 5\ zeros\ \ in\ |z| < 1$$

and

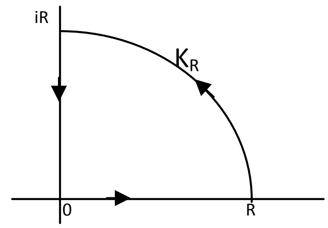
$$On\ |z| = 2,\ |z^7 + 4z^6 + 7z^3 + 2| < 442 < |-15z^5| = 482$$

$$\Rightarrow P\ has\ 5\ zeros\ in\ |z| < 2.$$

Therefore *P(z)* has no zeros in $1 < |z| < 2$.

Exercise. *Show that* $P(z) = z^5 + 2z^3 + 2z + 3$ *has exactly one root in first quadrant.*

Solution.



$$\Delta_{[0,R]}(\arg P(z)) = 0$$

$$\Delta_{K_R}(\arg P(z)) = \Delta_{K_R}[\arg(R^5 e^{5i\vartheta} + 2R^2 e^{2i\vartheta} + 2Re^{i\vartheta} + 3)]$$

$$= \Delta_{K_R}[5\vartheta] = \frac{5\pi}{2}.$$

$$\Delta_{[iR,0]}[\arg P(z)] = 0 - \frac{\pi}{2} = -\frac{\pi}{2}.$$

Therefore, $\frac{1}{2\pi}\Delta_{C_R}[\arg P(z)] = 1$, where $C_R = [0,R] \cup K_R \cup [iR,0]$.