Lecture 16

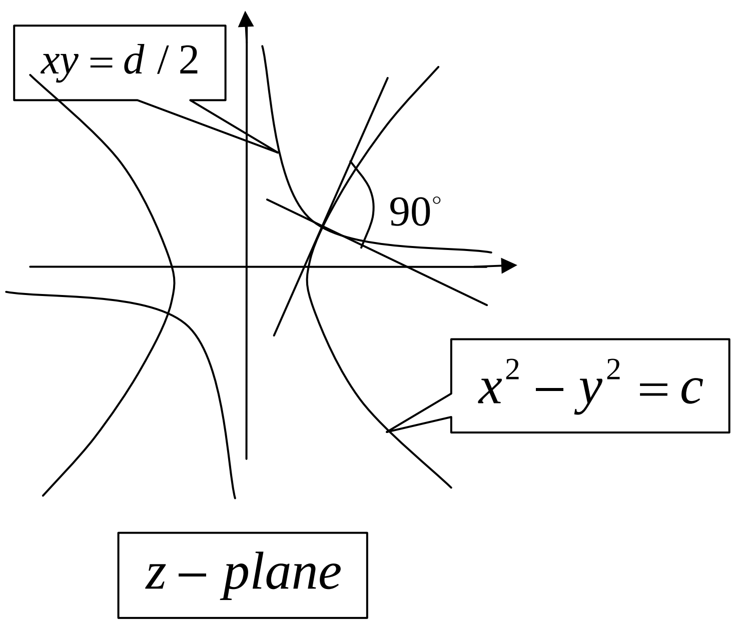
Mapping Properties of Analytic Functions

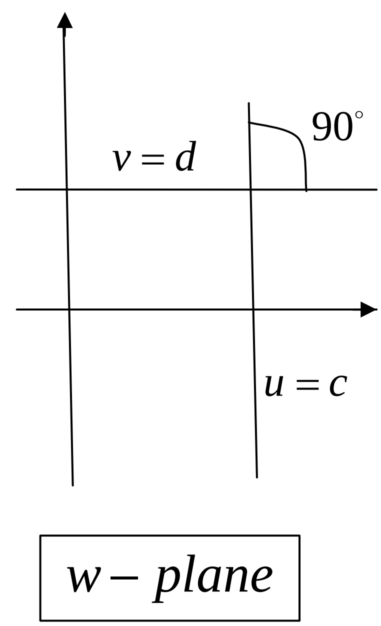
Example. Consider $f(z) = z^2$, $z = x + iy$ and $w = f(z) = u + iv$. Then

$$u = x^2 - y^2,\ v = 2xy \qquad (*)$$

Using (*), the following mapping properties of $f(z) = z^2$ are obtained:

(I) The hyperbolas $x^2 - y^2 = c$ *and* $2xy = d$ are mapped into straight lines $u = c$ *and* $v = d$. If $c, d \neq 0$, the hyperbolas as will as their images intersect at right angles as shown in the following figure:





Each branch of the hyperbola is mapped onto the same straight line. The image of the region $x > 0, y > 0,\ xy < 1$ is the strip $0 < v < 2$.

(II) The image of the lines $x=\pm c,(c\neq 0)$ is the same parabola

$$v^2=-4c^2(u-c^2),\left(\frac{dv}{du}\right)_1=-\frac{2c^2}{v}\ \text{.................}(1)$$

$$(since,\ x=\pm c\Rightarrow u=c^2-y^2\ and\ v=\pm 2yc\Rightarrow -y^2=u-c^2$$

$$and\ also\ y^2=\frac{v^2}{4c^2}\Rightarrow v^2=-4c^2(u-c^2))$$

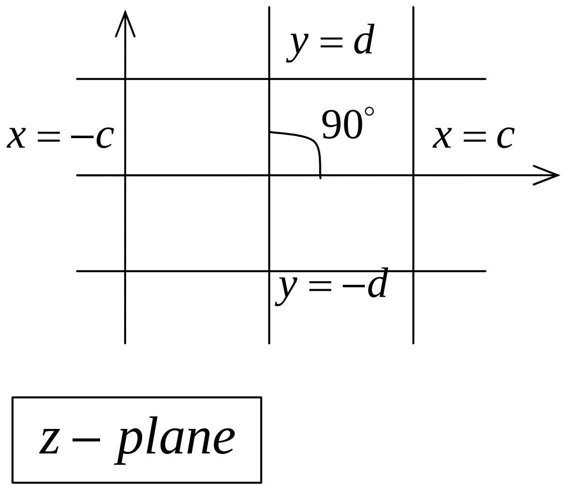
Similarly, the image of the lines $y=\pm d,(d\neq 0)$ is the same parabola

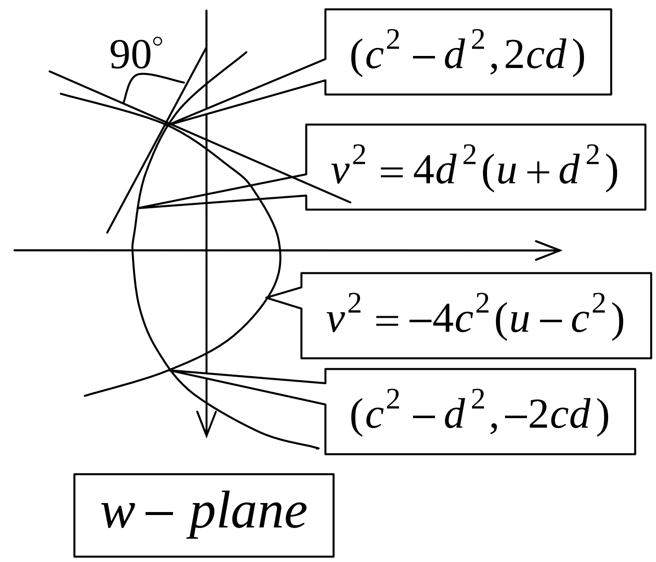
$$v^2=4d^2(u+d^2),\left(\frac{dv}{du}\right)_2=\frac{2d^2}{v}\ \text{...................}\ (2)$$

$$(since,\ y=\pm d\Rightarrow u=x^2-d^2\ and\ v=\pm 2xd\Rightarrow x^2=u+d^2$$

$$and\ also\ x^2=\frac{v^2}{4d^2}\Rightarrow v^2=4d^2(u+d^2))$$

Therefore, $\left(\frac{dv}{du}\right)_1\times\left(\frac{dv}{du}\right)_2=-\frac{4c^2d^2}{v^2}=-1\ at\ v=\pm 2cd$ implying that the lines $x=\pm c,\ y=\pm d$ as well as their images intersect at right angles as shown in the following figure:

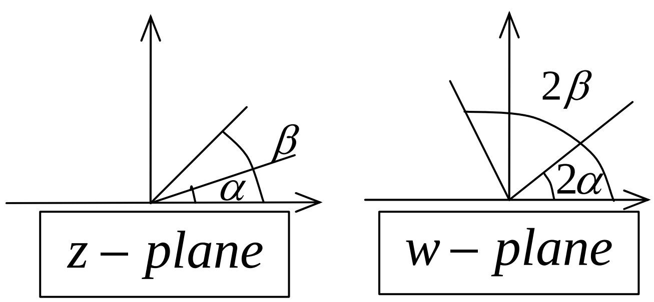

y = d
x = −c
90°
x = c
y = −d
z − plane


90°
$(c^2 - d^2, 2cd)$
$v^2 = 4d^2(u + d^2)$
$v^2 = -4c^2(u - c^2)$
$(c^2 - d^2, -2cd)$
w − plane

(III) If $z = re^{i\theta}$, $f(z) = r^2 e^{2i\theta}$. Thus, a circle of radius r is mapped onto a circle of radius r^2 in a 'one to two' fashion by $f(z) = z^2$.

(IV) Image of Sector $S(\alpha,\beta) = \{z : \alpha < \arg z < \beta\}$ is the sector $S(2\alpha,2\beta)$. Thus, the angle of the sector is doubled by the mapping $f(z) = z^2$

It also follows from the above, that the restriction of $f(z) = z^2$ to $S(\alpha,\beta)$ would be one to one only when $\beta - \alpha < \pi$.



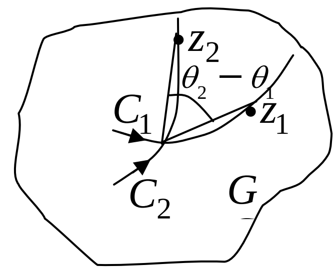
The observations in the above example are consequences of the following result:

Theorem. *Let f(z) be analytic in a domain G. Then, f(z) preserves angles between any two curves at a point* $a \in G$ *iff* $f'(a) \neq 0$.

If $f'(a) = \ldots = f^{(m-1)}(a) = 0,\ f^{(m)}(a) \neq 0$, *then angle between the curves at a are magnified m-times by the mapping f(z).*

Proof. Let $C_1 : z_1(t)$ *and* $C_2 : z_2(t),\ 0 \le t \le 1$ be any two curves in G that intersect at the point $a = z_1(t_1) = z_2(t_2)$. Let $z_1 \in C_1$ *and* $z_2 \in C_2$ be such that $|z_1 - a| = |z_2 - a| = r$. Then,

$$z_1 - a = re^{i\theta_1} \ \ and \ \ z_2 - a = re^{i\theta_2}$$



$$\Rightarrow \frac{z_1 - a}{z_2 - a} = e^{i(\theta_2 - \theta_1)} \Rightarrow \theta_2 - \theta_1 = \arg\left(\frac{z_2 - a}{z_1 - a}\right).$$

Let α be the angle between tangents to C_1 *and* C_2 at the point a. Then,

$$\alpha = \lim_{r \to 0} \arg\left(\frac{z_2 - a}{z_1 - a}\right)$$

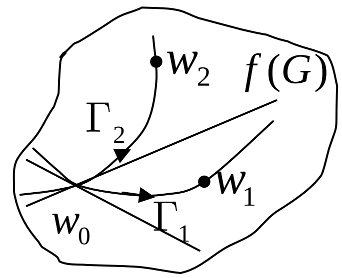
where, α is measured from arc $z_1(t)$ to arc $z_2(t)$.

Let $w_1 = f(z_1), w_2 = f(z_2)$ *and* $w_0 = f(a)$.

Further, let $\Gamma_1 = f(C_1)$ *and* $\Gamma_2 = f(C_2)$.

Then, the angle β between Γ_1 *and* Γ_2

is given by



$$\beta = \lim_{r\to 0} \arg\left(\frac{w_2 - w_0}{w_1 - w_0}\right), \text{ where}$$

$$w_2 - w_0 = \rho_2 e^{i\varphi_2} \text{ and } w_1 - w_0 = \rho_1 e^{i\varphi_1}.$$

Therefore, $$\beta = \lim_{r\to 0} \arg\left(\frac{f(z_2) - f(a)}{f(z_1) - f(a)}\right)$$

$$= \lim_{r\to 0} \arg\left[\left\{\frac{(f(z_2) - f(a))/(z_2 - a)}{(f(z_1) - f(a))/(z_1 - a)}\right\}\left\{\frac{z_2 - a}{z_1 - a}\right\}\right].$$

Now, since f is differentiable at the point a,

$$\lim_{r\to 0} \frac{f(z_2) - f(a)}{z_2 - a} = \lim_{r\to 0} \frac{f(z_1) - f(a)}{z_1 - a} = f'(a) \neq 0.$$

Therefore, $\beta = \lim_{r\to 0} \arg\left(\frac{z_2 - a}{z_1 - a}\right) = \alpha$ (*using continuity of the argument).* This also shows that the sense of rotation is also preserved.

Now, if $f'(a) = \ldots = f^{(m-1)}(a) = 0,\ f^{(m)}(a) \neq 0,$ then

$$f(z) = f(a) + c_m(z-a)^m + \ldots, \textit{where } c_m \neq 0.$$

Therefore, $\beta = \lim_{r\to 0} \arg\left\{\frac{f(z_2) - f(a)}{f(z_1) - f(a)}\right\} = \lim_{r\to 0} \arg\left\{\frac{z_2 - a}{z_1 - a}\right\}^m = m\alpha.$

$\Rightarrow$ The angle between image curves is magnified m-times.

Remarks.

(i) *Since* Γ_1 *has the parametric equation* $w_1(t) = f(z_1(t))$, $w_1'(t) = f'(z_1(t))\, z_1'(t)$, $z_1(t_0) = a$

$$\Rightarrow \arg w_1'(t_0) = \arg f'(z_1(t_0)) + \arg z_1'(t_0), \quad f'(a) \neq 0$$

$\Rightarrow$ *Curve* C_1 *at* a *rotates by an angle* $\arg f'(a)$ *and forms the curve* Γ_1 *at* $w_0 = f(a)$, *under the mapping f(z).*

(ii) *Further, since* $|f'(a)| = \lim_{z\to a} \dfrac{|f(z) - f(a)|}{|z-a|}$, *the expression* $|f'(a)|$ *gives the magnification in the image of the curve* C_1 *at the point a under the mapping f(z).*

(iii) *By (i) and (ii) above it follows that under the mapping f(z), at each point there is rotation by an amount* $\arg f'(a)$ *and magnification by an amount* $|f'(a)|$ *which is responsible in locally* ***distorting*** *the shape of the curve* C_1 *to its image* Γ_1, *under the mapping f(z).*

Conformal Mapping. *A mapping which preserves the angle as well as rotation between any two curves intersecting at the point a, is called a Conformal Mapping at the point a.*

Thus, if $f(z)$ is analytic at the point a and $f'(a) \neq 0$, then $f(z)$ is a conformal mapping at the point a.

Example1. $f(z) = \overline{z}$ *is not conformal at any point z.*

Example2. $f(z) = z^2$ *is not conformal at z = 0. It is conformal at any point* $z \neq 0$.