# Lecture 17

## ***A special class of Conformal Mappings: Mobius Transformations or Linear Fractional Transformations***

*A function S(z) of the form* $S(z)=\dfrac{az+b}{cz+d},\ \ ad-bc\neq 0$, *is called a* ***Mobius Transformation****.*

The complex numbers $a,b,c\ and\ d$ are called ***coefficients*** of Mobius Transformation *S(z)*.

The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called the ***determinant*** of Mobius Transformation *S(z).*

The constants $a,b,c\ and\ d$ do not uniquely determine S(z), since, f*or any* $\lambda\neq 0,\ S(z)=\dfrac{(\lambda a)z+(\lambda b)}{(\lambda c)z+(\lambda d)}$.

***Properties of Mobius Transformations:***

***1.*** The derivative of a Mobius Transformation $S(z)=\frac{az+b}{cz+d}$, $ad-bc\neq 0$, is nonzero for all $z\in\boldsymbol{C}-\{\frac{d}{c}\}$, *if* $c\neq 0$ *and all* $z\in\boldsymbol{C}$ *if* $c=0$**:**

$$\frac{d}{dz}S(z)=\frac{a(cz+d)-c(az+b)}{\left(cz+d\right)^2}=\frac{ad-bc}{\left(cz+d\right)^2}\neq 0,$$

for $z\in\boldsymbol{C}-\{\frac{d}{c}\}$, *if* $c\neq 0$ *and* $z\in\boldsymbol{C}$ *if* $c=0$.

Thus, every Mobius Transformation is a Conformal Mapping at every point $z\in\boldsymbol{C}-\{\frac{d}{c}\}$, *if* $c\neq 0$ *and* $z\in\boldsymbol{C}$ *if* $c=0$**.**

### *2. Composition of any two Mobius transformations is a Mobius transformation.*

It is easily seen that if

$$S_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \text{ and } S_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2},$$

with $a_1 d_1 - b_1 c_1 \neq 0$ and $a_2 d_2 - b_2 c_2 \neq 0$ then their composition

$$S^*(z) = S_1 \circ S_2(z) = \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_2^2)} = \frac{a^* z + b^*}{c^* z + d^*}$$

satisfies $a^* d^* - b^* c^* \neq 0$ and hence is a Mobius Transformation.

*(Hint: Observe that the determinant of $S^*(z)$ is the product of determinants of $S_1(z)$ and $S_2(z)$)*

**3. In general,** $S_1 \circ S_2(z) \neq S_2 \circ S_1(z)$

*Take, for example,* $S_1(z) = 1/z$ *and* $S_2(z) = z + 1$*.*

*Then,* $S_1 \circ S_2 = \frac{1}{z} + 1$, $S_2 \circ S_1 = \frac{1}{z+1}$, *so that* $S_1 \circ S_2 \neq S_2 \circ S_1$ .

***Exercise.*** *If* $S_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$ *and* $S_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$, *then*

$$S_1 \circ S_2(z) = \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_2^2)}$$

*and*

$$S_2 \circ S_1(z) = \frac{(a_2 a_1 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(c_2 a_1 + d_2 c_1) z + (c_2 b_1 + d_2 d_1)}$$

***4.*** $S(z)=\dfrac{az+b}{cz+d}$, $ad-bc\neq 0$, ***maps*** $\boldsymbol{C}\bigcup\{\infty\}$ ***one-one onto*** $\boldsymbol{C}\bigcup\{\infty\}$***.***

Note that $S(\infty)=\dfrac{a}{c}$, $S(-\dfrac{d}{c})=\infty$. Further, for any $w\in$ ***C****, there is a unique* $z\in$ ***C*** given by $z=\dfrac{dw-b}{-cw+a}$ . This proves $S(z)$ maps $C\bigcup\{\infty\}$ one-one onto $C\bigcup\{\infty\}$.

## 5. *Inverse of a Mobius Transformation is a Mobius Transformation*

The inverse of Mobius Transformation $S(z)=\dfrac{a\,z+b}{c\,z+d}$, $ad-bc\neq 0$, is $T(z)=\dfrac{d\,z-b}{-c\,z+a}$, since $S\circ T(z)=T\circ S(z)=z$. Obviously, $T(z)$ is a Mobius Transformation, since it's determinant is same as the determinant of $S(z)$ and hence is nonzero.

***6. A Mobius Transformation S(z) can have at most 2 fixed points (i.e. points $\alpha$ such that $s(\alpha)=\alpha$ ), unless S(z) ≡ z (i.e. $S \equiv I$ ).***

$S(z)=\dfrac{az+b}{cz+d}=z \Rightarrow cz^2+(d-a)z-b=0$*, which can have atmost two solutions.*

***7. A Mobius Transformation is uniquely determined by specifying its value at three distinct points.***

*Let a, b and c be three distinct points and S be a Mobius Transformation such that*

$$S(a) = \alpha,\ S(b) = \beta \text{ and } S(c) = \gamma.$$

Let T be another Mobius transformation such that

$$T(a) = \alpha,\ T(b) = \beta \text{ and } T(c) = \gamma.$$

*Then,* $T^{-1} \circ S(a) = a,\ T^{-1} \circ S(b) = b$ *and* $T^{-1} \circ S(c) = c$

$\Rightarrow T^{-1} \circ S$ *has three distinct fixed points.*

*Therefore, by Property 6 above,* $T^{-1} \circ S \equiv I \Rightarrow S \equiv T$.

***Elementary Mobius Transformations***

The Mobius Transformations

$$S(z) = z + a \ \textit{(translation)}, \quad S(z) = a\,z,\ a \neq 0,\ \textit{(dilation)}$$

$$S(z) = e^{i\theta} z \ \text{(rotation)},\ S(z) = \frac{1}{z} \ \text{(inversion)}$$

are called Elementary Mobius Transformations.

The properties of elementary Mobius Transformations are self-explanatory by their names.

***Proposition: Every Mobius Transformation is a composition of Translation, Dilation, Rotation and Inversion.***

*First let c = 0, so that* $S(z)=\frac{a}{d}z+\frac{b}{d}$. *Then,* $S=S_2\circ S_1$, *with* $S_2(z)=z+\frac{b}{d}$ *and* $S_1(z)=\frac{a}{d}z$.

*Next, let* $c\neq 0$ *and choose*

$$S_1(z)=z+\frac{d}{c}, S_2(z)=\frac{1}{z}, S_3(z)=\frac{bc-ad}{c^2}z \text{ and } S_4(z)=z+\frac{a}{c}$$

*Then,* $S=S_4\circ S_3\circ S_2\circ S_1$, *since*

$$S_4\circ S_3\circ S_2\circ S_1(z)=S_4\circ S_3\circ S_2(z+\frac{d}{c})$$

$$=S_4\circ S_3\left(\frac{1}{z+(d/c)}\right)=\frac{bc-ad}{c^2}.\frac{1}{z+\left(d/c\right)}$$

$$=\frac{bc-ad}{c(cz+d)}+\frac{a}{c}=\frac{bc-ad+acz+ad}{c(cz+d)}=\frac{c(az+b)}{c(cz+d)}=S(z).$$

***Proposition: Each elementary transformation maps a circle in*** $C\bigcup\{\infty\}$ ***to a circle in*** $C\cup\{\infty\}$ *(A circle in* $C\bigcup\{\infty\}$ *is defined to be a proper circle if it doesn't pass through* $\infty$ *and it's defined to be a straight line if it passes through* $\infty$*).*

*Any ordered triplet* $(z_1, z_2, z_3)$ *where* $z_i \in \Gamma,\ i = 1,2,3,$ *gives an orientation to the circle* $\Gamma$ *in* $C\bigcup\{\infty\}$**.** *For example,* $(1,\infty,-1)$ *and* $(1,-1,\infty)$ *give opposite orientations to the real line, while orientations* $(1,2,3)$ *and* $(1,\infty,-1)$ *give the same orientation to the real line.*

***Mappings of Proper Circle*** $C_r : |z - z_0| = r$ ***:***

| Transformation | Mapping of Cr |
|---|---|
| $w = z + a$ *or* $z = w - a$ | *Cr changes to* $\Gamma_r : \lvert w - a - z_0 \rvert = r$, *which is a circle with center* $a + z_0$ *and radius r* |
| $w = a\,z,$ $a \neq 0$ *or* $z = \dfrac{w}{a}$ | *Cr changes to* $\Gamma_r : \left\lvert \dfrac{w}{a} - z_0 \right\rvert = r$ *or* $\lvert w - az_0 \rvert = \lvert a \rvert r$, *which is a circle with center* $a\,z_0$ *and radius* $\lvert a \rvert r$. |

| | |
|---|---|
| $w = \frac{1}{z}$ *or* $z = \frac{1}{w}$ | <u>*Case (i)*</u>*(*$\left\lvert z_0 \right\rvert \neq r$*, i.e. circle does not pass through the origin): $C_r$ changes to* $\Gamma_r : \left\lvert \frac{1}{w} - z_0 \right\rvert = r \Rightarrow \frac{1}{\lvert w \rvert^2} - 2\,\mathrm{Re}(\frac{z_0}{\overline{w}}) + \lvert z_0 \rvert^2 - r^2 = 0$ <br> $\Rightarrow \lvert w \rvert^2 (\lvert z_0 \rvert^2 - r^2) - 2\,\mathrm{Re}\left(z_0 w_0\right) + 1 = 0$ <br> *which is a proper circle.* <br> <u>*Case( ii)*</u> *(*$\left\lvert z_0 \right\rvert = r$*, i.e. circle passes through the origin): $C_r$ changes to* $\Gamma_r : \left\lvert \frac{1}{w} - z_0 \right\rvert = r \Rightarrow \frac{1}{\lvert w \rvert^2} - 2\,\mathrm{Re}(\frac{z_0}{\overline{w}}) = 0$ <br> $\Rightarrow 2\,\mathrm{Re}(z_0 w) = 1,$ *which is a straight line* |

***Mapping of a straight line L :***

$\alpha x + \beta y = \gamma$ *or* $\mathrm{Re}(c\bar{z}) = \gamma$, *where* $z = x + iy$ *and* $c = \alpha + i\beta$**:**

| ***Transformation*** | ***Mapping of $C_r$*** |
|---|---|
| $w = z + a$ *or* $z = w - a$ | *L changes to* $\mathrm{Re}(c\bar{w} - c\bar{a}) = \gamma$ *or* $\mathrm{Re}(c\bar{w}) = \gamma + \mathrm{Re}(c\bar{a})$, *which is a straight line* |
| *w = a z,* $a \neq 0$ *or* $z = \frac{w}{a}$ | *L changes to* $\mathrm{Re}(\frac{c\bar{w}}{\bar{a}}) = \gamma$ *or* $\mathrm{Re}(ac\bar{w}) = \lvert a\rvert^2 \gamma$ *or* $\mathrm{Re}(d\bar{w}) = \lvert a\rvert^2 \gamma$, *where* $d = ac$ *, which is a straight line* |
| $w = \frac{1}{z}$ *or* $z = \frac{1}{w}$ | *L changes to* $\mathrm{Re}(\frac{c}{\bar{w}}) = \gamma$ *or* $\mathrm{Re}(cw) = \gamma \lvert w\rvert^2$, *which is a circle if* $\gamma \neq 0$ *and is a straight line if* $\gamma = 0$ |