Lecture 18

Example. Prove that the circles of Apollonius K: $\left|\frac{z-p}{z-q}\right| = k,\ k \neq 1$, are orthogonal to any circle C passing through the points $p\ and\ q$.

Consider the Mobius transformation $w = \frac{z-p}{z-q}$. It maps

(i) the circle K one-one onto the circle $|w| = k$

and

(ii) any circle C through the points $p\ and\ q$ is mapped one-one onto a straight line L passing through origin , since the points $p\ and\ q$ are mapped to the points 0 *and* ∞ by $w = \frac{z-p}{z-q}$.

Now, the circle $|w| = k$ is orthogonal to the Line L. Therefore, their images under the inverse of the transformation $w = \frac{z-p}{z-q}$ are Apollonius circle K and the circle C through the points $p\ and\ q$, also are orthogonal, since the inverse of the given Mobius transformation is also a Mobius transformation and hence preserves the angle of intersection.

Construction of a Mobius Transformation which takes the values 1, 0, ∞ at three given distinct points $z_2, z_3, z_4 \in \boldsymbol{C} \cup \{\infty\}$

Define, $S^*(z) = \dfrac{(z - z_3)}{(z - z_4)} / \dfrac{(z_2 - z_3)}{(z_2 - z_4)},\ z_2, z_3, z_4 \in \boldsymbol{C}$

$$= \frac{z - z_3}{z - z_4}, \quad if \ \ z_2 = \infty$$

$$= \frac{z_2 - z_4}{z - z_4}, \quad if \ \ z_3 = \infty$$

$$= \frac{z - z_3}{z_2 - z_3}, \quad if \ \ z_4 = \infty.$$

Then, $S^*(z_2) = 1, S^*(z_3) = 0, S^*(z_4) = \infty$. Moreover, S^* is the only Mobius Transformation having this property, since a Mobius Transformation is uniquely determined by specifying its action on three given distinct points.

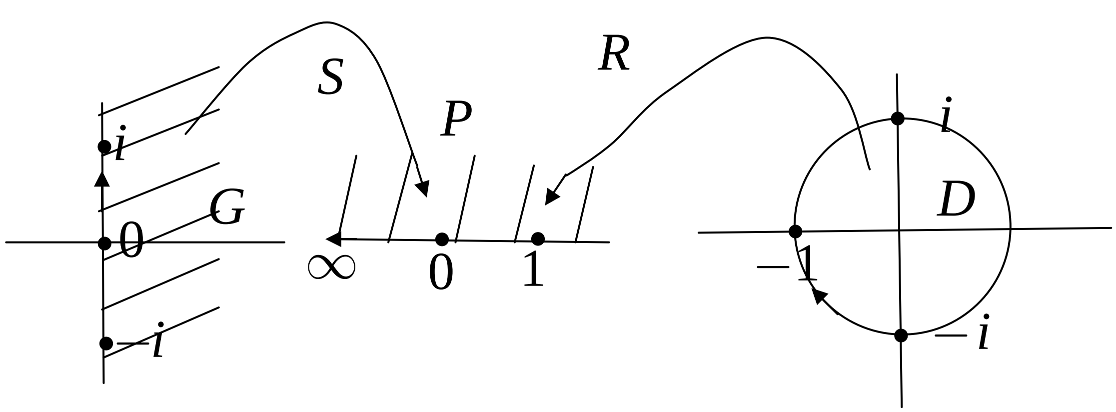
Notation: The transformation $S^*(z)$ is also denoted as (z, z_2, z_3, z_4)

Example. Construct a Mobius Transformation $T(z)$ which maps $G=\{z:\operatorname{Re} z>0\}$ onto $D=\{z:|z|<1\}$.

Solution. Give an orientation $(-i,\,0,\,i)$ to imaginary axis. Then, the Mobius Transformation that maps the imaginary axis onto real axis such that $-i\to 1,\ 0\to 0$ and $i\to\infty$ is given by

$$S(z)=(z,\,-i,\,0,\,i)=\frac{2z}{z-i}.$$

The region G lies to the right of the imaginary axis with respect to the orientation $(-i,\,0,\,i)$ of the imaginary axis. Further, the right hand side of real axis with orientation $(1,0,\infty)$ is the upper half plane P. Therefore, the image of G by Mobius Transformation S is upper half-plane P.



Next, let $\Gamma = \{z : |z| = 1\}$ *has the orientation*$(-i, -1,\ i)$. *Then, the Mobius Transformation that maps* Γ *onto real axis such that* $-i \to 1,\ -1 \to 0$ *and* $i \to \infty$ *is given by*

$$R(z) = (z,\ -i,\ -1,\ i) = \frac{2i}{i-1} . \frac{z+1}{z-i}.$$

Since, the disk D lies to the right of the circle Γ with orientation $(-i,\ -1,\ i)$ the Mobius Transformation R maps disk D on upper half- plane$P$, which lies to the right side of real line with orientation $(1,\ 0,\ \infty)$.

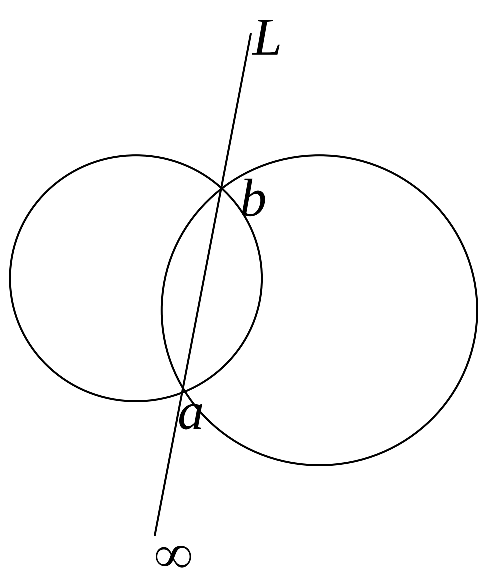
Consequently, $T(z) = \left(R^{-1} \circ S\right)(z) = \dfrac{z-1}{z+1}$ *is the required Mobius Transformation which maps* G *onto* D.

Example. Find the image of $\left\{z : |\operatorname{Im} z| < \frac{\pi}{2}\right\}$ under the mapping $g(z) = \tanh\frac{z}{2}$.

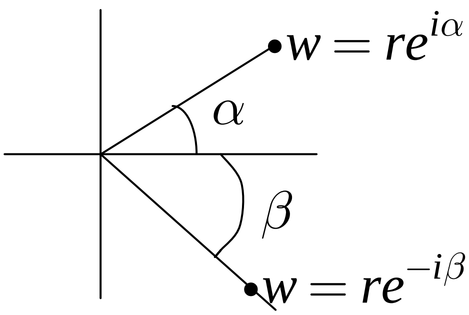
Solution: Observe that $g(z) = \tanh\frac{z}{2} = T(e^z) = \frac{e^z - 1}{e^z + 1}$, where T is the Mobius Transformation given by $T(z) = \frac{z-1}{z+1}$. Then, g(z) maps the infinite strip $\left\{z : |\operatorname{Im} z| < \frac{\pi}{2}\right\}$ onto $|z| < 1$, since e^z maps $\left\{z : |\operatorname{Im} z| < \frac{\pi}{2}\right\}$ onto $G = \{z : \operatorname{Re} z > 0\}$ and by the previous example $T(z) = \frac{z-1}{z+1}$ maps $G = \{z : \operatorname{Re} z > 0\}$ onto $|z| < 1$.

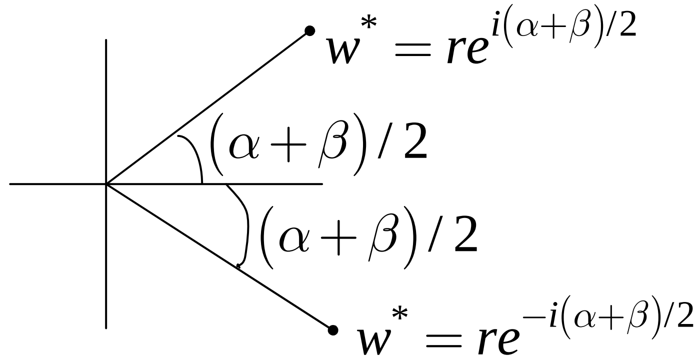
Example. Let G be the open region lying between the arcs of the circles Γ_1 *and* Γ_2 intersecting at the points a and b. Find a mapping of G onto a sector symmetric about the real axis.

Solution. Let L be the line passing through the points a and b. Give L the orientation (∞, a, b). Then the Mobius transformation T given by $T(z) = (z, \infty, a, b) = \dfrac{z-a}{z-b}$ maps L onto the real axis with $T(\infty) = 1, T(a) = 0, T(b) = \infty$. Now, since under a Mobius Transformation circles on C_∞ are mapped to circles in C_∞, images $T(\Gamma_1)$ *and* $T(\Gamma_2)$ are circles in C_∞ passing through 0 *and* ∞, i.e. these are straight lines.



Let, $T(G)=\{w:-\beta<\arg w<\alpha\}$, $\alpha,\beta>0$.





Now, to map the sector T(G) to a sector symmetric about the real axis, note that the opening angle of T(G) is $(\alpha+\beta)/2$. Therefore, if θ is the angle through which, by rotating the region $T(G)$, it becomes symmetric about real axis, then $\alpha+\theta=\dfrac{\alpha+\beta}{2}$, or $\theta=\dfrac{\beta-\alpha}{2}$. Note that $-\beta+\theta=-\dfrac{\alpha+\beta}{2}$. Such a Mobius Transformation is therefore $S(z)=e^{i\,(\beta-\alpha)/2}z$. Consequently, the required transformation is $S\circ T(z)=e^{i\,(\beta-\alpha)/2}\dfrac{z-a}{z-b}$.

Note. A transformation mapping the region G of above Example to the right half plane is given by

$$[S\circ T(z)]^{\pi/2(\alpha+\beta)}=\left[e^{i(\beta-\alpha)/2}\frac{z-a}{z-b}\right]^{\pi/2(\alpha+\beta)}$$