Lecture 3

Basic Properties of Differentiable Functions

Proposition 1. Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be defined in some neighbourhood of \( a \) and differentiable at \( a \). Then, \( \frac{\partial f}{\partial x}(a), \frac{\partial f}{\partial y}(a) \) exist and satisfy

\[
\frac{\partial f}{\partial x}(a) = -i \frac{\partial f}{\partial y}(a). \quad (*)
\]

Notes: 1. For a function \( \phi : \mathbb{C} \rightarrow \mathbb{R} \), the first order partial derivatives \( \phi_x \) (also denoted as \( \frac{\partial \phi}{\partial x} \)) and \( \phi_y \) (also denoted as \( \frac{\partial \phi}{\partial y} \)) at a point \( \zeta = (\alpha, \beta) \) are defined as, for \( t \in \mathbb{R} \),

\[
\phi_x = \lim_{t \to 0} \frac{\phi(\alpha + t, \beta) - \phi(\alpha, \beta)}{t}, \quad \phi_y = \lim_{t \to 0} \frac{\phi(\alpha, \beta + t) - \phi(\alpha, \beta)}{t}
\]

2. For a function \( f : \mathbb{C} \rightarrow \mathbb{C} \), given by \( f = u + i v \), the first order partial derivatives \( f_x \) and \( f_y \), at a point \( a = \alpha + i \beta \), are defined as, for \( t \in \mathbb{R} \),

\[
\begin{align*}
    f_x &= \lim_{t \to 0} \frac{f(\alpha + t, \beta) - f(\alpha, \beta)}{t} = u_x + i v_x, \\
    f_y &= \lim_{t \to 0} \frac{f(\alpha, \beta + t) - f(\alpha, \beta)}{t} = u_y + i v_y.
\end{align*}
\]
Proof of Proposition 1. Let $a = \alpha + i\beta$, $t \in \mathcal{R}$, $t \neq 0$. Then,

\[ f'(a) = \lim_{t \to 0} \frac{f(a + t) - f(a)}{t} = \lim_{t \to 0} \frac{f(\alpha + t, \beta) - f(\alpha, \beta)}{t} = \frac{\partial f}{\partial x}(a) \]

\[ f'(a) = \lim_{t \to 0} \frac{f(a + it) - f(a)}{it} = \lim_{t \to 0} \frac{f(\alpha, \beta + t) - f(\alpha, \beta)}{it} = -i \frac{\partial f}{\partial y}(a). \]

The identity (*) follows by the above identities.
**Proposition 2.** Let $f = u + i v$ be differentiable at $a \in \mathbb{C}$. Then,

$$
\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a), \quad \frac{\partial v}{\partial x}(a) = -\frac{\partial u}{\partial y}(a).
$$

**Proof.** By Prop. 1,

$$
\frac{\partial f}{\partial x}(a) = -i \frac{\partial f}{\partial y}(a)
$$

$$
\Rightarrow \frac{\partial u}{\partial x}(a) + i \frac{\partial v}{\partial x}(a) = -i(\frac{\partial u}{\partial y}(a) + i \frac{\partial v}{\partial y}(a))
$$

Equating real and imaginary parts of the above identity we get the result of Proposition 2.
Definition. Any one of the following equations

- \[ \frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y} \]
- \[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \]

are called \textit{Cauchy Riemann equations.}

Note that the above equations are equivalent.
**Example.** The function

\[
f(z) = \begin{cases} 
\frac{(z)^2}{z} & \text{if } z \neq 0 \\
0 & \text{if } z = 0 
\end{cases}
\]

satisfies CR equations at 0, but is not diff. at 0.

**f is not diff. at 0:**

\[
\lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{(\Delta z)^2} = \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}^2 \\
\quad \to 1 \quad \text{if } \Delta y = 0 \\
\quad \to 1 \quad \text{if } \Delta x = 0 \\
\quad \to \left(\frac{1-i}{1+i}\right)^2 \quad \text{if } \Delta x = \Delta y
\]
CR Equations are satisfied at 0:

Note that

\[ u(\Delta x, 0) = \text{Re} f(\Delta x, 0) = \text{Re} \Delta x = \Delta x \]
\[ u(0, \Delta y) = \text{Re} f(0, \Delta y) = \text{Re} \left(\frac{(i\Delta y)^2}{i\Delta y}\right) = \text{Re}(i \frac{\Delta y^2}{\Delta y}) = 0 \]
\[ v(\Delta x, 0) = \text{Im} f(\Delta x, 0) = \text{Im} \Delta x = 0 \]
\[ v(0, \Delta y) = \text{Im} f(0, \Delta y) = \text{Im}(i\Delta y) = \Delta y. \]

Consequently,

\[ \frac{u(\Delta x, 0) - u(0, 0)}{\Delta x} = \frac{\Delta x}{\Delta x} \to 1 \text{ as } \Delta x \to 0 \]
\[ \frac{u(0, \Delta y) - u(0, 0)}{\Delta y} = \frac{0}{\Delta y} \to 0 \text{ as } \Delta y \to 0 \]
\[ \frac{v(\Delta x, 0) - v(0, 0)}{\Delta x} = \frac{0}{\Delta x} \to 0 \text{ as } \Delta x \to 0 \]
\[ \frac{v(0, \Delta y) - v(0, 0)}{\Delta y} = \frac{\Delta y}{\Delta y} \to 1 \text{ as } \Delta y \to 0. \]

Therefore,

\[ \frac{\partial u}{\partial x}(0) = \frac{\partial v}{\partial y}(0) = 1; \frac{\partial v}{\partial x}(0) = -\frac{\partial u}{\partial y}(0) = 0. \]
Exercise. Prove that for the following functions CR equations are satisfied at 0 but the functions are not differentiable at 0:

(i) \[ f(z) = \begin{cases} \frac{z^5}{|z|^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \]

(ii) \[ f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \]
**Definition.** Any function \( \varphi : \mathbb{C} \rightarrow \mathbb{R} \) having continuous partial derivatives up to second order and satisfying the equation

\[
\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0
\]

(\(*\)) is called a **Harmonic Function or Potential Function.** Equation (\(*\)) is called **Laplace Equation** and \( \nabla^2 \varphi \) is called **Laplacian** of the function \( \varphi \).

Harmonic Functions are widely used in the study of steady state temperatures, wave theory, two dimensional electrostatics, fluid flow, robotics etc.

Since real and imaginary parts \( u \) and \( v \) of a complex differentiable function \( f = u + iv \) satisfy CR-equations, it easily follows that \( \nabla^2 u = 0, \nabla^2 v = 0 \). Thus, real and imaginary parts of a complex differentiable function are Harmonic functions.
The converse of Prop.1, proved below, holds under the additional hypothesis that $\partial f / \partial x, \partial f / \partial y$ exist, are continuous and satisfying CR equations $\partial f / \partial x = -i \partial f / \partial y$.

**Theorem.** Let, for $f(z)$ defined in a domain $G$, $\partial f / \partial x, \partial f / \partial y$ exist, are continuous and CR Equation $\partial f / \partial x = -i \partial f / \partial y$ is satisfied at any point $z_0 = (x_0, y_0) \in G$. Then, $f'(z_0)$ exists and is given by $f'(z_0) = u_x(x_0, y_0) + i v_y(x_0, y_0)$.

**Proof.** Let, $z_0 = x_0 + iy_0 \in G$ & $h = s + it$ be such that $|h| < \varepsilon$ and $z_0 + h \in G$.

Then,

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{u(x_0 + s, y_0 + t) - u(x_0, y_0)}{h} + i \frac{v(x_0 + s, y_0 + t) - v(x_0, y_0)}{h}$$  \hspace{1cm} (1)
Now,
\[
\frac{u(x_0 + s, y_0 + t) - u(x_0, y_0)}{h} = \frac{u(x_0 + s, y_0 + t) - u(x_0, y_0 + t)}{h} + \frac{u(x_0, y_0 + t) - u(x_0, y_0)}{h}
\]
\[
= \frac{\varphi(s, t)}{h} + \frac{s}{h} u_x(x_0, y_0) + \frac{t}{h} u_y(x_0, y_0)
\]
(\text{say, i.e. define } \varphi(s, t) \text{ by this identity})

Then,
\[
\varphi(s, t) = \frac{s}{h} u_x(x_0 + s_1, y_0 + t) + \frac{t}{h} u_y(x_0, y_0 + t_1)
\]

\[
- \frac{s}{h} u_x(x_0, y_0) - \frac{t}{h} u_y(x_0, y_0)
\]
\text{(By MVT & (2))}

\rightarrow 0 \text{ as } h \rightarrow 0 \quad (\because u_x \text{ & } u_y \text{ are continuous at } z_0 \text{ and } \left| \frac{s}{h} \right| \leq 1, \left| \frac{t}{h} \right| \leq 1)

Similarly,
\[
\frac{v(x_0 + s, y_0 + t) - v(x_0, y_0)}{h} = \frac{\psi(s, t)}{h} + \frac{s}{h} v_x(x_0, y_0) + \frac{t}{h} v_y(x_0, y_0),
\]
where, \[\frac{\psi(s, t)}{h} \rightarrow 0\]
\[ f(z_0 + h) - f(z_0) \]
\[ \frac{h}{f(z_0 + h) - f(z_0)} = \frac{s}{h} [u_x + i v_x] + \frac{t}{h} [u_y + i v_y] + \frac{\varphi + i \psi}{h} \]
\[ = \frac{s}{h} [u_x + i v_x] + \frac{t}{h} [-v_x + i u_x] + \frac{\varphi + i \psi}{h} \quad (\text{since } u_x = v_y \text{ & } u_y = -v_x) \]
\[ = \frac{s}{h} [u_x + i v_x] + \frac{it}{h} [u_x + i v_x] + \frac{\varphi + i \psi}{h} \]
\[ = u_x + iv_x + \frac{\varphi + i \psi}{h} \]
\[ \rightarrow u_x + iv_x \text{ as } h \rightarrow 0 \]
\[ \Rightarrow \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \text{ exists & equals } u_x + iv_x \text{ at } z_0. \]
Cauchy Riemann Equations in Polar Coordinates

Let $f(z) = u(r, \theta) + i v(r, \theta)$. Then,

$$ru_r = v_\theta, \quad rv_r = -u_\theta$$

are Cauchy Riemann equations in Polar coordinates.

The above equations can be easily obtained from CR equations in Cartesian coordinates as follows:

With $x = r \cos \theta$, $y = r \sin \theta$, 

$$u_r = u_x x_r + u_y y_r = u_x (\cos \theta) + u_y (\sin \theta)$$

$$v_\theta = v_x x_\theta + v_y y_\theta = v_x (-r \sin \theta) + v_y (r \cos \theta)$$

$\therefore ru_r = v_\theta$

by CR Equations $u_x = v_y$, $u_y = -v_x$ in cartesian coordinates.

Similarly, $rv_r = -u_\theta$.

The following proposition readily follows from the above theorem:
**Proposition 4.** Let \( f(z) = u(r, \theta) + iv(r, \theta) \). If \( \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta} \) exist, are continuous in some domain \( G \), and satisfy the following Cauchy Riemann equations in polar coordinates at a point \( z_0 = (r_0, \theta_0) \in G \),

\[
ru_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \\
r v_r(r_0, \theta_0) = -u_\theta(r_0, \theta_0)
\]

then, \( f'(z_0) \) exists and is given by

\[
f'(z_0) = \frac{u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)}{e^{i\theta_0}}.
\]

**Proof.** Since, \( r^2 = x^2 + y^2 \), \( \tan \theta = \frac{y}{x} \),

\[
u_x = u_r r_x + u_\theta \theta_x = u_r (\cos \theta) - u_\theta \frac{1}{r} (\sin \theta) \\
= u_r (\cos \theta) + v_r (\sin \theta)
\]

\[
u_x = v_r r_x + v_\theta \theta_x = v_r (\cos \theta) - v_\theta \frac{1}{r} (\sin \theta) \\
= v_r (\cos \theta) - u_r (\sin \theta)
\]

Therefore, using the above theorem,

\[
f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) \\
= \frac{u_r(r_0, \theta_0) + iv_r(r_0, \theta_0)}{e^{i\theta_0}}
\]
Elementary Functions

**Logarithmic Function.** Define

\[ \log z = \ln |z| + i \arg z \]

Since \( \arg z \) takes multiple values for every \( z \), \( \log z \) also takes multiple values for every \( z \) and is therefore not a function.

To make it a well-defined function, the range of \( \arg z \) has to be so restricted that it takes a unique value for every value of \( z \) in its domain of definition.

To this end, define

\[ \Log z = \ln |z| + i \Arg z \]

where, \( -\pi < \Arg z \leq \pi \). \( \Arg z \) takes a unique value for every \( z \neq 0 \). Consequently, \( \Log z \) is a well-defined function for every \( z \neq 0 \). \( \Log z \) is called the Principal Branch of Logarithmic Function.

- \( \Log z \) maps \( \mathbb{C} - \{0\} \) onto the strip \( \{ w : -\pi < \text{Im} z \leq \pi \} \).
Log \( z \) is continuous at all points in its domain of definition except at the points on the negative real axis, since \( \text{Arg} \, z \) is a continuous at all the points in its domain of definition, except the points on negative real axis.