Lecture 5

Power Series

A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called a power series. The complex numbers a_n's are called the ***coefficients*** and z_0 is called the ***centre*** of the power series.

For what values of z a power series converges?

To answer this question, we first review the definition and basic properties of lim sup & lim inf of a sequence $\{x_n\} \subseteq \boldsymbol{R}$.

Definition.

$$\lim \underset{(\text{inf})}{\sup}\; x_n = \underset{(\text{inf})}{\sup}\{set\ of\ all\ limit\ points\ of\ sequence\ \{x_n\}\}.$$

Basic Properties:

1. lim sup & lim inf always exist, these may possibly be $+\infty$ *or* $-\infty$.

2. lim sup & lim inf are unique

3. lim inf $x_n \leq$ lim sup x_n.

The following additional properties of lim sup & lim inf are used for derivation of the results concerning radius of convergence of a power series:

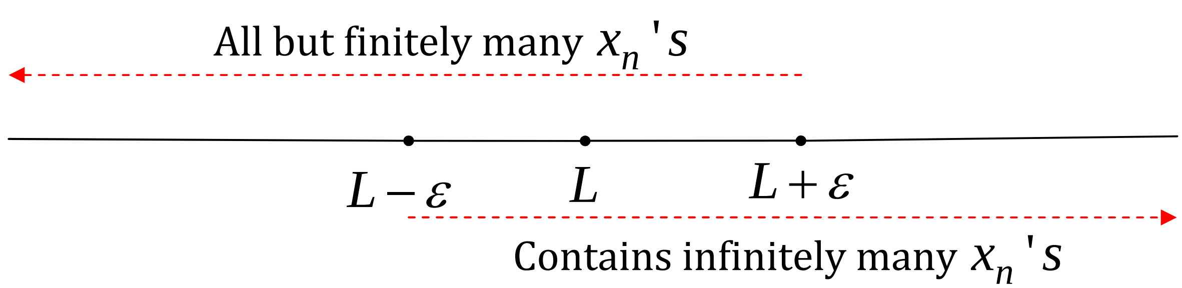
***Proposition* .** For any bounded sequence $\{x_n\} \subseteq \boldsymbol{R}$

$$\limsup_{n\to\infty} x_n = L \Leftrightarrow \textit{for any } \varepsilon > 0,\ x_n < L+\varepsilon \ \textit{ for all } n > n_0(\varepsilon)$$
$$x_{n_k} > L-\varepsilon \ \textit{ for some subsequence } \{x_{n_k}\} \textit{ of } \{x_n\}$$

$$\liminf_{n\to\infty} x_n = l \Leftrightarrow \textit{for any } \varepsilon > 0,\ x_n > l-\varepsilon \ \textit{ for all } n > n_0(\varepsilon)$$
$$x_{n_k} < l+\varepsilon \ \textit{ for some subsequence } \{x_{n_k}\} \textit{ of } \{x_n\}$$

Proof. We prove the theorem only for lim sup, the proof for lim inf can be constructed similarly.

(i) $\limsup x_n = L \Rightarrow L-\varepsilon \underset{\textit{for a subsequence } \{n_k\}}{<} x_n \underset{\textit{for all } n>n_0}{<} L+\varepsilon$



Suppose $x_n < L+\varepsilon$ *for all* $n > n_0$ is false.
$\Rightarrow x_n \geq L+\varepsilon_0$ *for infinitely many* n's *and* ε_0

$\Rightarrow$ *there exists a subsequence* $\{x_{n_k}\}$ *such that* $x_{n_k} \geq L + \varepsilon_0$ *for all* k

Since $\{x_{n_k}\}$ is a bounded sequence, it contains a convergent subsequence $\{x_{n_{k_l}}\}$. Let $x_{n_{k_l}} \to p_0$ *as* $l \to \infty$. Then, p_0 is a limit point of the sequence $\{x_n\}$.

By (*), $x_{n_{k_l}} \geq L + \varepsilon_0$ *for every* l

$$\Rightarrow p_0 \geq L + \varepsilon_0 \underset{\substack{(\text{since } L \text{ is } \sup \text{ of} \\ \text{all limit points})}}{\Rightarrow} L \geq L + \varepsilon_0 \Rightarrow \#.$$

('#' ≡ *Notation for 'a contradiction'*)

(ii) Suppose no subsequence $\{x_{n_k}\}$ can be found satisfying $x_{n_k} > L - \varepsilon_0$ for some ε_0.

$\Rightarrow$ *infinitely many* x_{n_k} *can never be greater than* $L - \varepsilon_0$

$\Rightarrow$ *every subsequence* $\{x_{m_k}\}$ *of* x_n *satisfies* $x_{m_k} \leq L - \varepsilon_0$ *for all* $k > k_0$

$\Rightarrow p \leq L - \varepsilon_0$ *for all limit points* $p \Rightarrow L \leq L - \varepsilon_0 \Rightarrow \#.$

Proposition. $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$ if and only if $\lim_{n\to\infty} x_n$ exists

****Proof.*** Let $\limsup_{n\to\infty} x_n = L$, $\liminf_{n\to\infty} x_n = L$ and $c = L = l$. Then, by Proposition 1,

$c - \varepsilon < x_n < c + \varepsilon \ \forall n > n_0 \Rightarrow \lim_{n\to\infty} x_n = c.$

Conversely, if $\lim_{n\to\infty} x_n = c$ exists, then the set of limit points of the sequence $\{x_n\}$ contains exactly one point c

$\Rightarrow L = l = c$.

Examples

1. $x_n = \begin{cases} 1 + \dfrac{1}{n} \ if \ \ n \ is \ even \\ -1 + \dfrac{1}{n} \ if \ \ n \ is \ odd \end{cases}$

For the sequence $\{x_n\}$, $\limsup x_n = 1$, $\liminf x_n = -1$.

2. For the sequence *{1,2,3,1,2,3,……………}*,

$x_{3n} = 3, x_{3n-1} = 2 \ and \ \ x_{3n-2} = 1$. Therefore,

$\limsup x_n = 3 \ and \ \liminf x_n = 1.$

Radius of Convergence.

For the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (1)$$

define a real number R by

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} \equiv L$$

and put $R = \infty$ *if* $L = 0$, $R = 0$ *if* $L = \infty$. The extended real number R is called the ***radius of convergence*** of the power series (1).

Note. The definition of radius of convergence can also be equivalently given as

$$R = \liminf_{n\to\infty} |a_n|^{-1/n} \qquad \textit{(prove!)}$$

The notion of radius of convergence easily describes all the points where (1) is convergent and all the points where (1) is not convergent.

Theorem 1. The power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely for all the points in $|z - z_0| < R$, is not convergent in $|z - z_0| > R$ and it converges uniformly in $|z - z_0| \le \rho < R$.

Proof.

(i) Let z be any arbitrary point in $|z - z_0| < R$. Assume that $|z - z_0| = r < R$. Let r_1 be such that $r < r_1 < R$.

$$\Rightarrow \frac{1}{r_1} > \frac{1}{R} = L.$$

By Proposition 1 on lim sup, $|a_n|^{1/n} < \frac{1}{r_1}$ for all $n > n_0$.

$$\Rightarrow \sum_{n=0}^{\infty} |a_n||z - z_0|^n = \sum_{n=0}^{\infty} |a_n| r^n < \sum_{n=0}^{\infty} (\frac{r}{r_1})^n \quad .$$

Since $\sum_{n=0}^{\infty} (\frac{r}{r_1})^n$ is cgt., by the comparison test $\sum_{n=0}^{\infty} |a_n||z - z_0|^n$ is convergent.

$\Rightarrow \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges absolutely in $|z - z_0| < R$.

(ii) If $|z-z_0| \le \rho$***,*** then $|a_n||z-z_0|^n < (\frac{\rho}{r_1})^n$, *where* $\rho < r_1 < R$*,* and uniform convergence follows.

(iii) Let z be any arbitrary point in $|z-z_0| > R$.
Let, $|z-z_0| = r > R$ and r_2 *be* such that $r > r_2 > R$.

$\Rightarrow \frac{1}{r_2} < \frac{1}{R} = L.$

By Proposition 1 on lim sup, there exists a subsequence $\{n_k\}$such that $|a_{n_k}|^{1/n_k} > \frac{1}{r_2}$.

$$\Rightarrow |a_{n_k}||z-z_0|^{n_k} = |a_{n_k}| r^{n_k} > (\frac{r}{r_2})^{n_k} \quad .$$

$\Rightarrow a_n(z-z_0)^n \nrightarrow 0$ as $n \to \infty. \Rightarrow \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is not cgt. in $|z-z_0| > R$.

Corollary. *If a power series* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at z = b, then it converges in
$|z-z_0| < |b-z_0|$.

The following theorem gives that the function represented by a power series is analytic in its disk of convergence:

Theorem 2. *If the radius of convergence of the power series* $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ *is R, then*

(a) *radius of convergence of the series*

$$\sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)\ a_n (z - z_0)^{n-k} \qquad (*)$$

is also R for every k = 1, 2, ...

(b) *Define f by f(z) =* $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, *then f is infinitely many times differentiable in* $|z - z_0| < R$ *.*

(c) $\dfrac{f^{(k)}(z_0)}{\underline{|k}} = a_k, \quad k = 1, 2, \dots$ **.**

Proof.

Without loss of generality assume that $z_0 = 0$.

(a) Let radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ be R and radius of convergence of $\sum_{n=0}^{\infty} n a_n z^{n-1}$ be R'. We prove $R = R'$. The result for general k follows by induction.

Since, $\overline{\lim_{n\to\infty}} |n a_n|^{1/(n-1)} = \overline{\lim_{n\to\infty}} | a_n|^{1/(n-1)}$, radius of convergence of $\sum_{n=0}^{\infty} a_n z^{n-1}$ is also R'. Now, $|a_0| + \sum_{n=1}^{\infty} |a_n||z|^n = |a_0| + |z| \sum_{n=1}^{\infty} |a_n||z|^{n-1}$.

Series on RHS cgs in $|z| < R' \Rightarrow$ Series on LHS cgs in $|z| < R' \Rightarrow R \geq R'$
Series on LHS cgs in $|z| < R \Rightarrow$ Series on RHS cgs in $|z| < R \Rightarrow R' \geq R$.

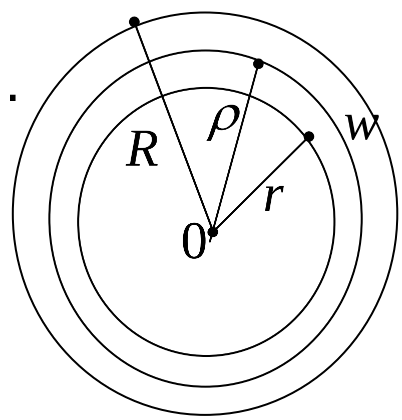
Therefore, $R' = R$.

(b) In $|z| < R$, define $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. We prove:
given $\varepsilon > 0, \exists\, \delta > 0$ *such that*
$\left|\frac{f(z) - f(w)}{z - w} - g(w)\right| < \varepsilon$ *whenever* $|z - w| < \delta$.

Write, for all n,

$$\frac{f(z) - f(w)}{z - w} - g(w) = \left[\frac{S_n(z) - S_n(w)}{z - w} - S_n'(w)\right] + [S_n'(w) - g(w)] + \left[\frac{R_n(z) - R_n(w)}{z - w}\right] \qquad (1)$$

where $S_n(z) = \sum_{k=0}^{n} a_k z^k$ *and* $R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$.



Let $|w| = r < R$. Then,

$$\left|\frac{R_n(z) - R_n(w)}{z - w}\right| \le \sum_{k=n+1}^{\infty} |a_k| \left|\frac{z^k - w^k}{z - w}\right|$$

Choose ρ such that $r < \rho < R$, so that

$$\frac{z^k - w^k}{z - w} = \left|z^{k-1} + z^{k-2} w + \ldots + z^2 w^{k-2} + w^{k-1}\right|$$

$$< k\rho^{k-1} \quad in\ |z - w| < \delta_1 \subseteq |z| < \rho$$

$$\Rightarrow \left|\frac{R_n(z) - R_n(w)}{z - w}\right| \le \sum_{k=n+1}^{\infty} k\, |a_k| \rho^{k-1}$$

$$< \varepsilon / 3 \quad \forall n > n_1(\varepsilon) \ \ and \ \ |z - w| < \delta_1 \qquad (2)$$

Next,

$$\lim_{n\to\infty} S_n'(w) = g(w) \Rightarrow \left|S_n'(w) - g(w)\right| < \varepsilon/3 \quad for\ \ \forall n > n_2(\varepsilon) \qquad (3)$$

Let $n_0 = \max(n_1, n_2)$. Take $N > n_0$. Choose $\delta_2 > 0$ such that

$$\left|\frac{S_N(z) - S_N(w)}{z - w} - S_N'(w)\right| < \varepsilon/3 \ \ for\ \ 0 < |z - w| < \delta_2 \subseteq |z| < \rho \qquad (4)$$

Write (1) as

$$\frac{f(z) - f(w)}{z - w} - g(w) = [\frac{S_N(z) - S_N(w)}{z - w}] + [S_N'(w) - g(w)] + [\frac{R_N(z) - R_N(w)}{z - w}]$$

which, in view of (2), (3) and (4) implies that

$$\left|\frac{f(z) - f(w)}{z - w} - g(w)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\ \forall z\ in\ 0 < |z - w| < \delta = \min(\delta_1, \delta_2)$$

$\Rightarrow\ f'(w) = g(w)$

$\Rightarrow\ f'(z)$ is given by a series of the form (*) with $k = 1$ and $z_0 = 0$.

Since (*) with $k=1$ has radius of convergence R, the above arguments give that $f''(w)$ exists and is given by $\sum_{n=2}^{\infty} n(n-1)a_n(z-z_0)^{n-2}$.

An induction argument gives that $f^{(k)}(z)$ exists in $|z|<R$ for all k = 1, 2, 3,...., and is given by (*) with $z_0=0$.

(c) Since $f^{(k)}(z)$ is given by a series of the form (*), put $z = z_0$ in (*) to give $a_k = \frac{f^{(k)}(z_0)}{\underline{|k}}$