Lecture 6

To Find Radius of Convergence From Ratio of Consecutive Terms $|a_n / a_{n-1}|$ of the Power Series $\sum_{n=0}^{\infty} a_n (z-z_0)^{\lambda_n}$, $a_n \neq 0$ for all n.

The formula for radius of convergence in terms of $|a_{n-1} / a_n|$ does not work if $a_n = 0$ for infinitely many n's.

The Series $\sum_{n=0}^{\infty} a_n (z-z_0)^{\lambda_n}$ with $a_n \neq 0$ for all n is called a ***Power Series with Gaps****, if $\lambda_n \neq n$.*

Theorem 3.

If $\lim_{n\to\infty} |a_n / a_{n-1}|^{1/(\lambda_n - \lambda_{n-1})}$ exists, then $\frac{1}{R} = \lim_{n\to\infty} |a_n / a_{n-1}|^{1/(\lambda_n - \lambda_{n-1})}$, where R is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^{\lambda_n}$, $a_n \neq 0$ for all n and $\{\lambda_n\}$ is any increasing sequence of non-negative integers such that $\lambda_n \to \infty$ *as* $n \to \infty$.

Proof. Let $\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left|a_n / a_{n-1}\right|^{1/(\lambda_n - \lambda_{n-1})} = \begin{matrix}B^*\\ A^*\end{matrix}$ ***.*** *Then,*

$$(A^* - \varepsilon)^{\lambda_n - \lambda_{n-1}} < |a_n / a_{n-1}| < (B^* + \varepsilon)^{\lambda_n - \lambda_{n-1}} \text{ for all } n > n_0 .$$

$$\Rightarrow \left|a_{n_0}\right|(A^* - \varepsilon)^{\lambda_n - \lambda_{n_0}} < \left|a_n\right| < \left|a_{n_0}\right|(B^* + \varepsilon)^{(\lambda_n - \lambda_{n-1}) + \ldots\ldots + (\lambda_{n_0+1} - \lambda_{n_0})}$$

$$\left(\text{since } \left|a_n\right| = \left|a_{n_0}\right| \left|\frac{a_{n_0+1}}{a_{n_0}}\right| \ldots. \left|\frac{a_n}{a_{n-1}}\right|\right)$$

$$\Rightarrow \liminf_{n\to\infty} \left|a_n\right|^{1/\lambda_n} \geq A^* \text{ and } \limsup_{n\to\infty} \left|a_n\right|^{1/\lambda_n} \leq B^*.$$

Since, $\lim_{n\to\infty} \left|a_n / a_{n-1}\right|^{1/(\lambda_n - \lambda_{n-1})}$ *exists,* $A^* = B^*$.

$$\Rightarrow \lim_{n\to\infty} \left|a_n\right|^{1/\lambda_n} \text{ exists and } \frac{1}{R} = \lim_{n\to\infty} \left|a_n\right|^{1/\lambda_n} = \lim_{n\to\infty} \left|a_n / a_{n-1}\right|^{1/(\lambda_n - \lambda_{n-1})}$$

Example. $\sum_{n=0}^{\infty} \frac{1}{2^n}(z - z_0)^{n^2}$ **.**

The radius of convergence R of the above power series is given by

$$\frac{1}{R} = \lim_{n\to\infty}\left(\frac{2^{n-1}}{2^n}\right)^{1/(n^2-(n-1)^2)}$$

$$= \lim_{n\to\infty}\left(\frac{1}{2}\right)^{1/(2n+1)} = 1$$

Radius of Convergence of Product of Power Series

Let

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \textit{ has radius of convergence } R_1$$

and

$$\sum_{n=0}^{\infty} b_n (z - z_0)^n \textit{ has radius of convergence } R_2$$

***Hadamard Product*:** $\sum_{n=0}^{\infty} a_n b_n \left(z - z_0\right)^n$ is called the Hadamard Product of the above two power series.

Let R^* be its radius of convergence.

Since, $\limsup_{n\to\infty} |a_n b_n|^{1/n} \le \limsup_{n\to\infty} |a_n|^{1/n} \limsup_{n\to\infty} |b_n|^{1/n}$ (prove!)

$$\Rightarrow \frac{1}{R^*} \le \frac{1}{R_1 R_2} \quad \Rightarrow R^* \ge R_1 R_2 \ge [\min(R_1, R_2)]^2$$

***Cauchy Product*:**

The power series $\sum_{n=0}^{\infty}(a_0 b_n + \dots + a_n b_0)\,(z - z_0)^n$ (*)

is called the *Cauchy Product* of the above two Power series. Let R be its radius of convergence.

Proposition. *If the radius of convergence of*

$f(z) = \sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n$ *is* R_1 *and radius of convergence of*

$g(z) = \sum_{n=0}^{\infty} b_n \left(z - z_0\right)^n$ *is* R_2*, then the radius of convergence of their Cauchy product is* $R = \min(R_1, R_2)$*.*

Proof. WLOG assume that $z_0 = 0$.

Let $S_n(z)$ *be* n^{th} *partial sum* $\sum_{k=0}^{n} a_k z^k$ (1)

$T_n(z)$ *be* n^{th} *partial sum* $\sum_{k=0}^{n} b_k z^k$ (2)

$P_n(z)$ *be* n^{th} *partial sum* $\sum_{k=0}^{n}(a_0 b_k + \dots + a_k b_0) z^k$.

To show: If f(z) is limit of (1) as $n \to \infty$ and g(z) is limit of (2) as $n \to \infty$, then (*) has the sum *f(z).g(z)* in $\left|z - z_0\right| < R = \min(R_1, R_2)$, where R_1 is the radius of convergence for power series of *f(z) and* R_2 *is the radius of convergence for power series of g(z).*

Now,

$$P_n(z) = a_0b_0 + (a_0b_1 + a_1b_0)z + ... + (a_0b_n + ... + a_nb_0)z^n$$

$$= a_0T_n(z) + a_1T_{n-1}(z)z + ... + a_nT_0(z)z^n$$

$$= a_0(\varepsilon_n(z) + g(z)) + a_1(\varepsilon_{n-1}(z) + g(z))z + ... + a_n(\varepsilon_0(z) + g(z))z^n,$$

where $\varepsilon_n(z) = T_n(z) - g(z) \Rightarrow T_n(z) = \varepsilon_n(z) + g(z)$

$$= [S_n(z)g(z)] + [a_0\varepsilon_n(z) + a_1\varepsilon_{n-1}(z)z + ... + a_n\varepsilon_0(z)z^n].$$

Since, $\varepsilon_n(z) \to 0$ *as* $n \to \infty$ *in* $|z| < R_2$, $|\varepsilon_n(z)| < \varepsilon \ \forall n > N$ and further $\alpha(z) = \sum_{n=0}^{\infty} |a_n||z|^n < \infty$ *in* $|z| < R_1$,

$$|\gamma_n(z)| = |a_0\varepsilon_n(z) + a_1z\varepsilon_{n-1}(z) + ... + a_nz^n\varepsilon_0(z)|$$

$$\le |a_nz^n\varepsilon_0(z) + ... + a_{n-N}z^{n-N}\varepsilon_N(z)|$$

$$+ |a_{n-(N+1)}z^{n-(N+1)}\varepsilon_{N+1}(z) + ... + a_0\varepsilon_n(z)|$$

$$< |a_nz^n\varepsilon_0(z) + ... + a_{n-N}z^{n-N}\varepsilon_N(z)|$$

$$+ \varepsilon\alpha(z) \qquad in \ |z| < \min(R_1, R_2)$$

$$\Rightarrow \lim_{n\to\infty} |\gamma_n(z)| \le \varepsilon\alpha(z) \qquad (since \ a_nz_n \to 0 \ as \ n \to \infty)$$

$$\Rightarrow \lim_{n\to\infty} \gamma_n(z) = 0, \qquad (since \ \varepsilon \ is \ arb.)$$

$$\Rightarrow \lim_{n\to\infty} P_n(z) = \lim_{n\to\infty} S_n(z)g(z) = f(z).g(z).$$

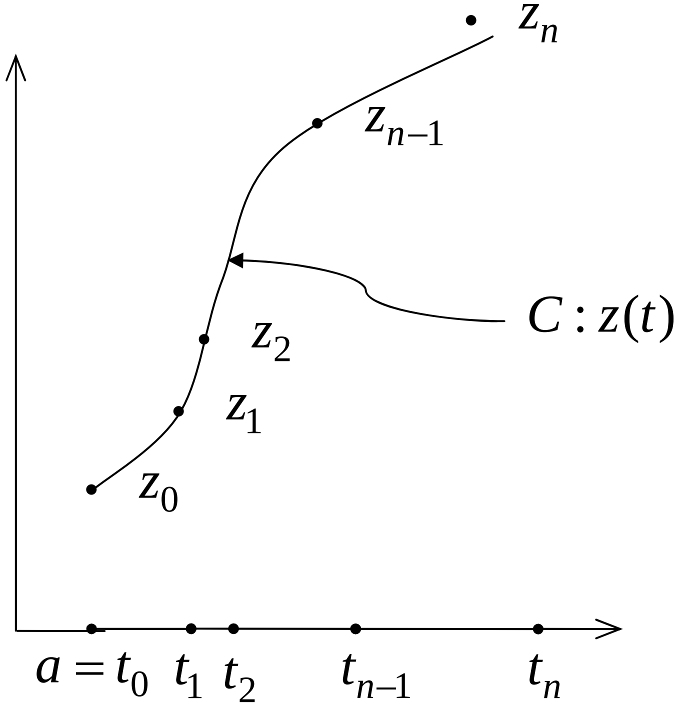
Complex Integration

Let $C : z(t),\ a \le t \le b$, be a continuously differentiable curve, i.e. *z(t)* is a continuously differentiable function in $[a,b]$.

For any partition

$\{a = t_0, t_1, \ldots, t_{m-1}, t_m = b\}$ of $[a,b]$,

let $z(t_j) = z_j$, $j = 0,1,\ldots,m$



Let the function $f : \boldsymbol{C} \to \boldsymbol{C}$ be continuous on the curve C.

Consider the sum $S_n = \sum_{m=1}^{n} f(\varsigma_m)(z_m - z_{m-1})$

where, ς_m is any point on the curve lying between z_{m-1} and z_m.

Definition. Complex Integration of f on C *is defined as*

$$\int_C f(z)\,dz = \lim_{n\to\infty} S_n,\ \textit{provided}\ \max_{1\le m\le n} |\Delta z_m| \to 0\ \textit{as}\ n \to \infty$$

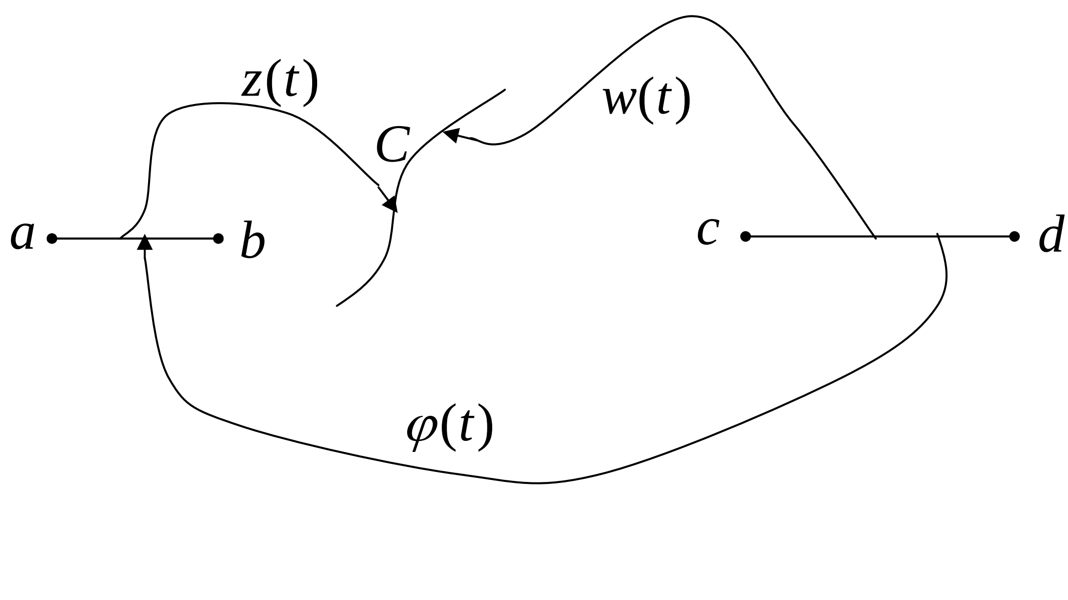
Equivalently, using the definition of integral of real functions,

$$\int_C f(z)\, dz = \int_a^b f(z(t))\, \dot{z}(t)\, dt. \qquad (*)$$

Note. *The definition (*) is independent of the parametric representation of C. For if w(t) ,* $c \le t \le d$*, is another parametric representation of C . Let* $\varphi : [c,d] \to [a,b]$ *be one-one onto differentiable function such that* $\varphi(c) = a,\ \varphi(d) = b$ *and* $w(t) = z(\varphi(t))$*. Then,*

$$\int_c^d f(w(t))\, \dot{w}(t)\, dt = \int_c^d f(z(\varphi(t)))\, \dot{z}(\varphi(t))\dot{\varphi}(t)\, dt$$

$$= \int_{\varphi(c)}^{\varphi(d)} f(z(x))\, \dot{z}(x)\, dx = \int_a^b f(z(x))\, \dot{z}(x)\, dx$$



For studying the properties of integration of a function $f : \boldsymbol{C} \to \boldsymbol{C}$, we need the definition and properties of a function $F : [a,b] \to \boldsymbol{C}$.

Integration of Functions $F : [a,b] \to \boldsymbol{C}$

Let $F(t) = u(t) + i\,v(t)$, $a \le t \le b$. Define,

$$\int_a^b F(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Properties.

(i) $\operatorname{Re}\int_a^b F(t)\,dt = \int_a^b \operatorname{Re} F(t)\,dt$

(ii) $\int_a^b \gamma F(t)\,dt = \gamma \int_a^b F(t)\,dt$, $\gamma = a$ *complex constant*

(iii) $\left|\int_a^b F(t)\,dt\right| \le \int_a^b |F(t)|\,dt$

Proof: (i) and ***(ii)*** immediately follow from the definition.

(iii) Let $r_0 e^{i\theta_0} = \int_a^b F(t)\,dt$. Then,

$$r_0 = \int_a^b e^{-i\theta_0} F(t)\ dt$$

$$= \int_a^b \operatorname{Re}\left(e^{-i\theta_0} F(t)\right) dt \qquad (using\ (i))$$

$$\le \int_a^b \left|e^{-i\theta_0} F(t)\right|\ dt = \int_a^b |F(t)|\ dt \quad (using\ property\ of\ real\ integral)$$

Properties of Complex Integration $\int_C f(z)\,dz$

(1) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$ (use that if parametric rep. of *C is* $z(t),\ a \le t \le b$*, then parametric rep. of −C is* $z(-t): -b \le t \le -a$ *or, alternatively,* $z(b+(a-t)): a \le t \le b$*)*

(2) $\int_C \gamma f(z)\,dz = \gamma \int_C f(z)\,dz$ (*easily follows from definition*)

(3)
$\int_C (f+g)\,dz = \int_C f\,dz + \int_C g\,dz$ (*follows easily from definition*)

(4) If C_1 is continuous curve from α_1 *to* β_1 *,* C_2 is a continuous curve α_2 *to* β_2 *and* $\beta_1 = \alpha_2$ and the curve C is union of curves C_1 and C_2, then

$$\int_C f(z)\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz$$

(5) $\left|\int_C f(z)\,dz\right| \le ML$, *where L is the length of* C *and* $|f(z)| \le M$ *for* $z \in C$ *.*

*(The property (5) is called **ML-Estimate** of the integral).*

Proof (4):

Let $C_1 : z_1(t),\ 0 \le t \le 1$ *and* $C_2 : z_2(t),\ 0 \le t \le 1$.

Then, *C = z(t), where* $z(t) = \begin{cases} z_1(2t), & 0 \le t \le 1/2 \\ z_2(2t-1), & 1/2 \le t \le 1 \end{cases}$

Therefore,

$$\int_C f(z)\,dz = \int_0^1 f(z(t))\dot{z}(t)\,dt$$

$$= \int_0^{1/2} f(z(t))\dot{z}(t)\,dt + \int_{1/2}^1 f(z(t))\dot{z}(t)\,dt$$

$$= \int_0^{1/2} f(z_1(2t))(\dot{z}_1(2t).2)\,dt + \int_{1/2}^1 f(z_2(2t-1))(\dot{z}_2(2t-1).2)\ dt$$

$$= \int_0^1 f(z_1(t)(\dot{z}_1(t).)\,dt + \int_0^1 f(z_2(t)\,\dot{z}_2(t)\ dt$$

$$= \int_{C_1} f(z)\ dz + \int_{C_2} f(z)\ dz$$

Proof (5):

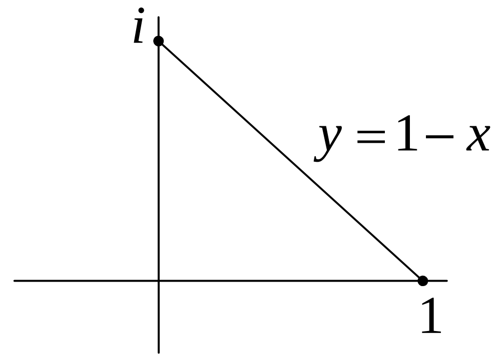
$$\left|\int_C f(z)\,dz\right| \le \int_a^b \left|f(z(t))\right|\left|\dot{z}(t)\right| dt$$

$$\le M\int_a^b \left|\dot{z}(t)\right| dt = ML$$

Example 1: Show that

$\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$, *where* C : *The line segment joining* i *and* 1***.***

Solution:



L = length of C = $\sqrt{2}$

On C, $|z| = \sqrt{x^2 + y^2} = \sqrt{x^2 + (1-x)^2}$

$$\Rightarrow |z|^4 = (x^2 + (1-x)^2)^2 = (2x^2 - 2x + 1)^2 = [2(x - \tfrac{1}{2})^2 + \tfrac{1}{2}]^2 \ge \frac{1}{4}$$

Using Property 5, we now get the required estimate of the integral in Example 1.