Lecture 7

Cauchy Theorem:

Let $f$ be analytic inside and on a simple, closed, piecewise smooth curve $C$. Then,\
\[
\int_C f(z) \, dz = 0.
\]

Definitions: Let $z(t)$, $a \leq t \leq b$, be parametric representation of the curve $C$.

Simple Curve: The curve $C$ is said to be simple, if it does not have any self-intersections (i.e. $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$ ($a < t_1, t_2 < b$)).

Closed Curve: The curve $C$ is said to be Closed, if end point of the curve is the same as its initial point (i.e. $z(a) = z(b)$).

Piece-wise smooth Curve: The curve $C$ is said to be Piece-wise smooth, if $z(t)$ is piece-wise differentiable (i.e. differentiable for all except finitely many $t$) and $\frac{d}{dt} z(t)$ (denoted as $\dot{z}(t)$) is piece-wise continuous in the interval $[a, b]$.
Proof (Under the assumption that \( f'(z) \) is continuous on \( C \))

By Green’s Theorem,

\[
\oint_{C} P \, dx + Q \, dy = \iint_{R} \left( Q_x - P_y \right) \, dx \, dy,
\]

where, curve \( C \) is boundary of the region \( R \) and the first partial derivatives \( P, Q, Q_x, P_y \) exist and are continuous in \( C \cup R \).

The hypothesis of Cauchy Theorem implies that the conditions of Green’s Theorem are satisfied.

Now, \( \int_{C} f(z) \, dz = \int_{a}^{b} f(z(t)) \, \dot{z}(t) \, dt \)

\[
= \int_{a}^{b} (u + iv)(\dot{x}(t) + i\dot{y}(t)) \, dt
\]

\[
= \int_{a}^{b} (u\dot{x} - v\dot{y}) \, dt + i \int_{a}^{b} (u\dot{y} + v\dot{x}) \, dt
\]

\[
= \int_{C} ud\alpha - v\,d\beta + i \int_{C} u\,d\beta + v\,d\alpha
\]

\[
= -\iint_{R} \left( u_y + v_x \right) \, dx \, dy + i \iiint_{R} \left( u_x - v_y \right) \, dx \, dy
\]

\[
= 0
\]
The proof of Cauchy Theorem in the general case, where the continuity of $f'(z)$ is not assumed, is beyond the scope of this course.
Cauchy Theorem for Multiply Connected Domains (Domain with holes).

Let simple closed piece-wise smooth curves $C_1,\ldots,C_n$ be enclosed by a simple, closed piece-wise smooth curve $C$, all the curves being oriented anticlockwise. Let $D$ be domain with boundary curves $C, C_1,\ldots,C_n$ (Such a domain is called a multiply connected domain). If a function $f(z)$ is analytic on $D \cup C \cup C_1 \cup \ldots \cup C_n$, then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz.$$
**Proof:** Join $C$ (oriented anticlockwise) and $C_1^*, ..., C_n^*$ (the curves $C_1, ..., C_n$ oriented clockwise) by straight line segment as shown in the figure for $n = 3$. Observe that with these orientations, $D$ lies to left if one traverses along any of these curves.

Applying Cauchy Theorem to the simply connected domain bounded by the curve

$$
\Gamma = L_1 \cup C_1^{*u} \cup L_2 \cup C_2^{*u} \cup L_3 \cup C_3^{*u} \cup ... \cup L_n \cup C_n^* \cup
$$

$$
- L_n \cup C_{n-1}^{*l} \cup ... - L_3 \cup C_2^{*l} \cup - L_2 \cup C_1^{*l} - L_1 \cup C
$$

where, $C_i^{*u}$ denotes the upper part of the curve $C_i^*$ and $C_i^{*l}$ denotes the lower part of the curve $C_i^*$(observe that $\Gamma$ has positive orientation, since the domain bounded by it lies to its left when one traverses on $\Gamma$), it follows that
\[ \oint_{C} f(z) \, dz + \oint_{C_1^*} f(z) \, dz + \ldots + \oint_{C_n^*} f(z) \, dz = 0 \]

(since the integrals along \( L_i \)'s are equal and opposite to each other)

\[ \Rightarrow \oint_{C} f(z) \, dz = \oint_{C_1} f(z) \, dz + \ldots + \oint_{C_n} f(z) \, dz \]

**Corollary.** If \( f \) is analytic (i) on two simple, closed, piece-wise smooth curves \( C_1 \) and \( C_2 \) and (ii) inside the domain bounded by \( C_1 \) and \( C_2 \), then

\[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz. \]

The above corollary helps in evaluation of integrals over curve \( C_1 \), the determination of whose parametric representation may be complicated. In such a case, the possibility of obtaining a curve \( C_2 \) satisfying the conditions of the corollary and whose parametric representation is simple to obtain, is explored and the integral is evaluated with the help of above corollary.
**Example:** Evaluate $\oint_{\Gamma} \frac{1}{w-z_0} \, dw$, where $\Gamma$ is any anticlockwise oriented simple closed piecewise smooth curve and $z_0$ is a point lying in the bounded domain $D$ with boundary $\Gamma$.

Note that direct evaluation of the above integral is not possible, since any explicit equation of $\Gamma$ is not known. However, this integral could be simply evaluated by using the above theorem.

Consider any anticlockwise oriented circle $C_r : |w-z_0| = r$, with $r$ small enough so that $C_r$ lies in $D$. The function $\frac{1}{w-z_0}$ is analytic on the curves $\Gamma$ and $C_r$ and in the domain bounded by these curves. Therefore, by Cauchy Theorem for Multiply connected domains,

$$\oint_{\Gamma} \frac{1}{w-z_0} \, dw = \oint_{C_r} \frac{1}{w-z_0} \, dw = \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} \, dt = 2\pi i$$

since, $w(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$, is a parametric representation of the circle $C_r$. 
Cauchy Integral Formula: If \( f \) is analytic in a domain \( G \) and \( \overline{B(a,r)} \subseteq G, \) where \( \overline{B(a,r)} = \{w : |w-a| \leq r\}. \) Then, for any \( z \in \{|w-a| < r\} \)

\[
f(z) = \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{(w-z)} \, dw
\]

(1)

where, \( C_r : w(t) = z + re^{it}, \, 0 \leq t \leq 2\pi. \)

Proof: Consider a circle \(|w-z| = \delta^*\) centered at \( z \) and having radius \( \delta^* \) sufficiently small such \(|w-z| = \delta^* \) \( \subseteq \{|w-a| < r\}. \) Then, by Cauchy Theorem of Multiply Connected Domains,

\[
\oint_{C_r} \frac{f(w)}{(w-z)} \, dw = \oint_{|w-z|=\delta^*} \frac{f(w)}{(w-z)} \, dw
\]

since the integrand is an analytic function in the domain lying between \( C_r \) and \(|w-z| = \delta^*\). Now, note that

\[
\oint_{|w-z| = \delta^*} \frac{f(w)}{(w-z)} \, dw = \oint_{|w-z| = \delta^*} \frac{f(w) - f(z)}{(w-z)} \, dw + f(a) \oint_{|w-z| = \delta^*} \frac{1}{(w-z)} \, dw
\]

(*)

The second term of (*) = \(2\pi i f'(z)\). Therefore, Cauchy Integral Formula follows if we prove that the first term of (*) is zero.

For this use continuity of \( f(w) \) at 'z', which gives that for every \( \varepsilon > 0, \) there exists \( \delta > 0 \) such that \(|f(w) - f(z)| < \varepsilon \) whenever \(|w-z| < \delta\). Choose \( \delta^* < \delta\).
Then,

\[ \left| \int_{|w-z|=\delta^*} \frac{f(w) - f(z)}{(w-z)} \, dw \right| < \frac{\varepsilon}{\delta^*} \times 2\pi \delta^* = 2\pi \varepsilon \quad \text{(by ML-Estimate)} \]

\[ \Rightarrow \int_{|w-z|=\delta^*} \frac{f(w) - f(z)}{(w-z)} \, dw = 0 \quad \text{since } \varepsilon \text{ is arbitrary.} \]

**Note:** In view of Cauchy Theorem for multiply connected domains, Cauchy Integral Formula (1) remains valid with \( C_r \) replaced by any simple closed piece-wise smooth curve \( \Gamma \) so that (i) every point enclosed by \( \Gamma \) is in \( D \) (ii) \( \Gamma \) encloses the point \( z \). This is because the function \( f'(w)/(w-z) \) is analytic in the domain lying between \( C_r \) and \( \Gamma \).