Lecture 8

*We show that every analytic function can be expanded into a power series, called the **Taylor series** of the function.*

Taylor's Theorem: *Let f be analytic in a domain D & $a \in D$. Then, f(z) can be expressed as the power series*

$$f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n \qquad (1)$$

where, $b_n = \dfrac{1}{2\pi i} \oint_{C_r} \dfrac{f(w)}{(w-a)^{n+1}} dw = \dfrac{f^{(n)}(a)}{n!}, \; f^{(0)}(a) \equiv f(a),$

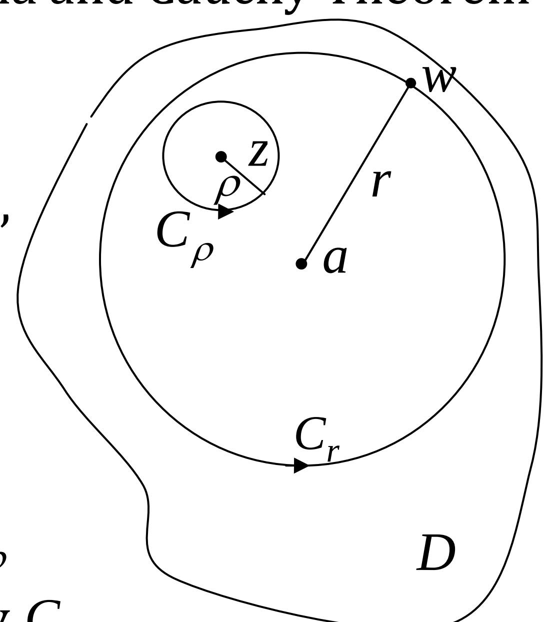
where, $C_r \subset D$ is *a counterclockwise oriented circle, of radius* r *and center at* a, *such that it encloses only points of* D.

The representation (1) is unique and is valid in the largest open disk with center a, contained in D.

Proof: By using Cauchy Integral Formula and Cauchy Theorem For Multiply Connected Domains,

$$f(z) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{w-z} dw,$$

where, z is any point enclosed by the circle C_r and C_ρ is a counterclockwise oriented circle $|w-z| = \rho$ with sufficiently small radius ρ such that C_ρ lies in the bounded domain enclosed by C_r.



Now,

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \frac{1}{w-a}[1-\frac{z-a}{w-a}]^{-1}$$

Recall that,

$$1+q+\ldots+q^n = \frac{1-q^{n+1}}{1-q}$$

$$\Rightarrow \frac{1}{1-q} = 1+q+\ldots+q^n + \frac{q^{n+1}}{1-q},$$

for any complex number q

Let $q = \dfrac{z-a}{w-a}$. Then,

$$\frac{1}{w-z} = \frac{1}{w-a}[1+\frac{z-a}{w-a}+\ldots+(\frac{z-a}{w-a})^n + \frac{1}{w-a}\frac{(\frac{z-a}{w-a})^{n+1}}{1-\frac{z-a}{w-a}}]$$

$$\Rightarrow \frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-a}dw + (z-a)\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{(w-a)^2}dw + \ldots$$

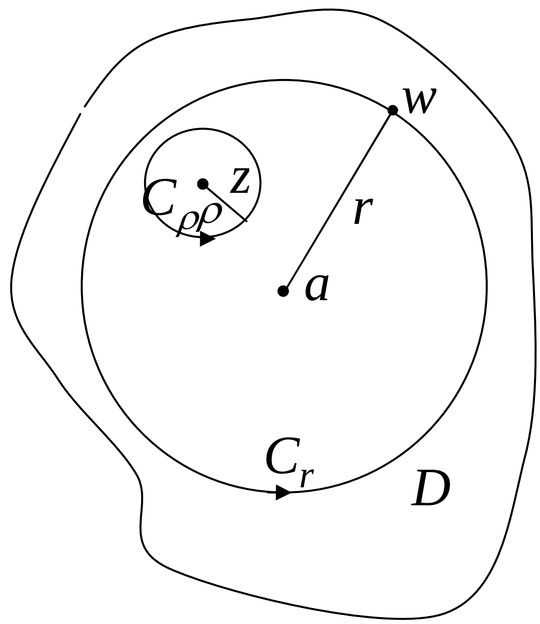
$$\ldots + (z-a)^n\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{(w-a)^{n+1}}dw$$

$$\underbrace{+(z-a)^{n+1}\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{(w-a)^{n+1}(w-z)}dw}_{R_n(z)}$$

where,

$$\left|R_n(z)\right| < \frac{\left|z-a\right|^{n+1}}{2\pi} \cdot \frac{M^*(r)}{r^{n+1}} .2\pi r, \quad for \quad M^*(r) = \max_{w \in C_r} \left| \frac{f(w)}{w-z} \right|$$

$$= r\,M^*(r) \left|\frac{z-a}{r}\right|^{n+1} \to 0 \text{ as } n \to \infty, \quad since \left|\frac{z-a}{r}\right| < 1.$$



Thus, $f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$,

with $b_n = \dfrac{1}{2\pi i} \oint_{C_r} \dfrac{f(w)}{(w-a)^{n+1}} dw$.

Further, since $f(z)$ is represented by power series, by a previous proposition on power series, $f(z)$ is infinitely many times differentiable in $\left|z-a\right| < r$ and

$$b_n = \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{(w-a)^{n+1}} dw = \frac{f^{(n)}(a)}{n!}.$$

Since $b_n = \dfrac{f^{(n)}(a)}{n!}$, it depends only on f *and* 'a', so b_n's are uniquely determined.

(because, if $f(z) = \sum_{n=0}^{\infty} b_n^* (z-a)^n$, $b_n^* = \dfrac{f^{(n)}(a)}{n!} = b_n$).

Thus, (1) represents *f* uniquely.

Proposition: *Every function $f(z)$, analytic in a domain D, is infinitely many times differentiable in D.*

Proof: $D = \bigcup_{a \in D} \{|z-a| < r_a\}$.

- By Taylor's Theorem, for every $a \in D$, $f(z)$ is represented by a power series in $|z-a| < r_a$.
- By an earlier proposition on power series, the functions represented by a power series are infinitely many times differentiable.

So that $f(z)$ is infinitely many times differentiable in $|z-a| < r_a$ for every $a \in D$.

Therefore, $f(z)$ is infinitely many times differentiable in D.

Cauchy Integral Formula for nth-derivative

If f *is analytic in a domain* D *and* $\overline{B(a,r)} \subseteq D$*, where* $\overline{B(a,r)} = \{w : |w-a| \le r\}$. T*hen,*

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{C_r} \frac{f(w)}{(w-a)^{n+1}} dw, \ \ n = 0,1,2,\ldots \quad (*)$$

where, $C_r : w(t) = a + re^{it}$, $0 \le t \le 2\pi$, is *a counterclockwise oriented circle of radius* r *centred at* a.

Proof: Follows immediately since, by the proof of Taylor's Theorem, $b_n = \frac{1}{2\pi i} \oint_{C_r} \frac{f(w)}{(w-a)^{n+1}} dw = \frac{f^{(n)}(a)}{n!}$.

For $n = 0$, denoting $f^{(0)}(a) \equiv f(a)$, (*) becomes **Cauchy Integral Formula.**

Note: In view of Cauchy Theorem for multiply connected domains, formula (*) remains valid with C_r replaced by any simple closed piece-wise smooth curve Γ so that (i) every point enclosed by Γ is in D (ii) Γ encloses the point a. This is because the function $f(w)/(w-a)^{n+1}$ is analytic in the domain lying between C_r and Γ.

Remark: *The formula (*) gives the value of the function and its derivatives at any point enclosed by a simple closed piecewise differentiable curve* Γ*, if the values of the function on* Γ *are known.*

This helps in knowing the values of the function and its derivatives at sometimes inaccessible points through values at accessible points.

A Computational Method, called ***Complex Variable Boundary Element Method****, developed using (*), is a great tool to computationally generate the values of* $f(a), f'(a), f''(a), \ldots$ *etc..*

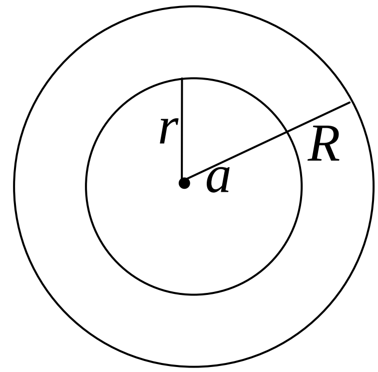
Deductions From Tayolor's Theorem:

Proposition 1: *Every power series with nonzero radius of convergence is the Taylor series of the function represented by it.*

Proof: Let (*) $\sum_{n=0}^{\infty} b_n (z-a)^n$ represents the function *f(z)* in $|z-a|<R$, i.e. $f(z)=\sum_{n=0}^{\infty} b_n (z-a)^n \ in \ |z-a|<R$. Then, by the proof of Taylor's Theorem, $b_n = \dfrac{f^{(n)}(a)}{n!}$. This implies that the given series (*) is the Taylor series of *f.*

Proposition 2 (***Cauchy's Estimate***): Let f be analytic and $\left|f(z)\right| \le M(R)$ *on* $\left|z-a\right| < R$. Then,

$$\left|f^{(n)}(a)\right| \le \frac{n!M(R)}{R^n}.$$



Proof: By Cauchy Integral Formula for n$^{\text{th}}$-derivative (Take $D = \{|z-a| < R\}$, for any $r < R$,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{C_r} \frac{f(w)}{(w-a)^{n+1}}dw,\ \ n = 0,1,2,\ldots$$

$$\Rightarrow \left|f^{(n)}(a)\right| \le \frac{n!}{2\pi}\frac{M(R)}{r^{n+1}}.2\pi r = \frac{n!M(R)}{r^n} \text{ (using ML-Estimate)}$$

Since $r < R$ is arbitrary, the result follows on letting $r \to R$.

***Proposition 3* (*Liouville's Theorem*):** An entire (*i.e. analytic in the whole Complex Plane*) function that is bounded in the whole Complex Plane is constant.

Proof: Since f is entire and bounded in the whole complex plane, $|f(z)| \le M$ on every circle $C_R \equiv \{z : |z| = R\}$.

Now, expand $f(z)$ in to Taylor series as $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for z in $|z| < R_0$. The same expansion is valid for $|z| \le R \;\; for \;\; all \;\; R > R_0$.

By Cauchy Estimate,

$$\Rightarrow |a_n| = \left| \frac{f^n(0)}{n!} \right| \le \frac{M}{R^n} \to 0 \; as \; R \to \infty, \quad for \; all \; n = 1, 2 \ldots$$

$\Rightarrow f(z) \equiv a_0 = constant, \; on$ every disk $|z| \le R$

Consequently $f(z)$ is constant in the whole complex plane ***C***, since $R > R_0$ is arbitrary.

***Proposition 4* (*Fundamental Theorem of Algebra*):**

A polynomial of degree n has exactly n complex zeros (counted according to multiplicity).

Proof: Let $P_n(z)$ be a polynomial of degree $n \geq 1$. and it has no zeros in the complex plane ***C***. Then, the function $\varphi(z) = \dfrac{1}{P_n(z)}$ (i) is an entire function (ii) is bounded in ***C*** (*since* $P_n(z) \to \infty$ as $z \to \infty$).

Therefore, by Liouville's Theorem, $\varphi(z)$ is constant.
$\Rightarrow$ $P_n(z)$ is also a constant function, a contradiction.

Thus, $P_n(z)$ has at least one zero, say a_1 of multiplicity m_1.

Now, the polynomial $\dfrac{P_n(z)}{(z-a_1)^{m_1}}$, is of degree $n - m_1$. A repetition of the above arguments gives that it has at least one zero, say a_2 of multiplicity m_2.

Continuing the process, it follows that $P_n(z)$ has $m_1 + m_2 + \ldots + m_k = n$ zeros at $a_1, a_2, \ldots, a_k$.

Proposition 5. *If f is an entire function and* $|f(z)| \le MR^{n_0}$ *in* $|z| \le R$ *for every* R, $0 \le R < \infty$ *then f is a polynomial of degree at most* n_0.

Proof: By Taylor's Theorem, expand $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < R_0$. The same expansion is valid for all $R > R_0$.

By Cauchy Estimate,

$$\left| f^{(n)}(0) \right| \le \frac{n!M(R)}{R^n}, \text{ where } M(R) = \max_{|z|=R} |f(z)|$$

$$\therefore |a_n| \le \frac{MR^{n_0}}{R^n} = MR^{n_0 - n} \to 0 \text{ as } n \to \infty, \text{ if } n > n_0.$$

$\Rightarrow$ f is a polynomial of degree at most n_0.