Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MSO202A/MSO202 Quiz 1 Grading Scheme (August 17, 2013)
Introduction To Complex Analysis

Roll No.: ………………. Section:………… **Time: 40 Minutes**
Marks: 40

Name: ……………………………………………..

Note: Give only answers (no details of workout) for Problems 1 to 8 at dotted lines. Each of these problems is of 5 marks.

- Put a cross mark at the wrong answer and tick mark at the correct answer.
- Where part marks are awarded, indicate the missing/wrong part of the answer.

1. If $(-3)^{i\sqrt{2}} = a + i\,b; a, b\ real$, then

   $(i)\ a =$ ……………………………………………… $(ii)\ b =$ ………………………………………………

Solution:

$$(-3)^{i\sqrt{2}} = \exp(i\sqrt{2}\log(-3))$$
$$= \exp\{\sqrt{2}(i\ln 3 - (2n+1)\pi)\}$$
$$= \exp(-\sqrt{2}\,(2n+1)\pi)\{\cos(\sqrt{2}\ln 3) + i\sin(\sqrt{2}\ln 3)\}$$

$\Rightarrow (i)\ a = \exp(-\sqrt{2}\,(2n+1)\pi)\cos(\sqrt{2}\ln 3)$ $\quad (ii)\ b = \exp(-\sqrt{2}\,(2n+1)\pi)\sin(\sqrt{2}\ln 3)$

(3 marks for one correct, 5 marks for both correct)

2. Let complex numbers $z_k,\ 1 \le k \le 5,$ be the roots of the equation $z^5 = 32\,i$. Then, values of $Arg\ z_k$ in the interval $[0, 2\pi]$ are given by

   $(i)\ Arg\ z_1 =$ ………… $(ii)\ Arg\ z_2 =$ ………… $(iii)\ Arg\ z_3 =$ ………… $(iv)\ Arg\ z_4 =$ ………… $(v)\ Arg\ z_5 =$ …………

Solution:

$$z_k = 2e^{i(\frac{\pi}{2}+2k\pi)/5} = 2e^{i(\frac{\pi}{10}+\frac{2k\pi}{5})}$$

$$\Rightarrow Arg\ z_1 = \frac{\pi}{10}, Arg\ z_2 = \frac{\pi}{10}+\frac{2\pi}{5}, Arg\ z_3 = \frac{\pi}{10}+\frac{4\pi}{5}, Arg\ z_4 = \frac{\pi}{10}+\frac{6\pi}{5},\ Arg\ z_4 = \frac{\pi}{10}+\frac{8\pi}{5}.$$

(5 marks, deduct 1 mark for each wrong/missing answer)

3. The Cartesian equation of boundary curve of the region $\{z : \mathrm{Re}\{\frac{z(2z+i)}{2z-i}\} > 0,\ \mathrm{Re}\, z < 0\}$ is

……………………………………………………………………………………..

Solution: $\mathrm{Re}\{\frac{z(2z+i)}{2z-i}\} > 0 \Leftrightarrow \mathrm{Re}\{\frac{z(2z+i)(2\bar{z}+i)}{|2z-i|^2}\} \Leftrightarrow \mathrm{Re}\{z(4|z|^2 + 4i\,x - 1)\} > 0$

$\Leftrightarrow x(4x^2 + 4y^2 - 4y - 1)\} > 0$

$\Rightarrow$ *For* $z = x + iy,\ x < 0,$ the region is $4x^2 + 4y^2 - 4y - 1 < 0$. Therefore, the required boundary is the semicircle $4x^2 + 4y^2 - 4y - 1 = 0,\ x < 0$

(5 marks, deduct 1 marks if the condition $x < 0$ is not mentioned in the answer)

4. The region in which the function

$f(z) = \left|\mathrm{Re}\, z^2\right| + i\left|\mathrm{Im}\, z^2\right|$

is analytic, is $D_1 \cup D_2 \cup D_3 \cup D_4$. Then,

(i) $D_1 = \{z = re^{i\theta}$: ……………………}   (i) $D_2 = \{z = re^{i\theta}$: ……………………}

(i) $D_3 = \{z = re^{i\theta}$: ……………………}   (i) $D_4 = \{z = re^{i\theta}$: ……………………}

Solution: Observe that $f(z) = \left|x^2 - y^2\right| + 2i\,\left|xy\right|$ can be written as

$f(z) = z^2$ *for* $0 < \theta < \pi/4$ *and* $\pi < \theta < 5\pi/4$,

$f(z) = -\bar{z}^2$ *for* $\pi/4 < \theta < \pi/2$ *and* $5\pi/4 < \theta < 3\pi/2$,

$f(z) = -z^2$ *for* $\pi/2 < \theta < 3\pi/4$ *and* $3\pi/2 < \theta < 7\pi/4$,

$f(z) = \bar{z}^2$ *for* $3\pi/4 < \theta < \pi$ *and* $7\pi/4 < \theta < 2\pi$.

Consequently, the function is analytic in the regions

$0 < \theta < \pi/4,\ \pi < \theta < 5\pi/4,\ \pi/2 < \theta < 3\pi/4,\ 3\pi/2 < \theta < 7\pi/4.$

Further, along the rays $\theta = 0, \pi/4,\ \pi/2,\ 3\pi/4,\ \pi,\ 5\pi/4,\ 3\pi/2,\ 7\pi/4,$ either the real part or the imaginary part of f(z) is zero, so it is not analytic on these rays.

(5 marks, deduct 2 marks if rays are also included in the region of analyticity)

5. Let the function $f(z)=u(x,y)+i\,v(x,y)$ be defined by

$$f(z)=\begin{cases}\dfrac{(1+\frac{i}{2})\,z^5}{|z|^4}, & if\;\; z\neq 0\\ 0, & if\;\; z=0\,.\end{cases}$$

Then,

(i) $u_x(0,0)=$ .... (ii) $u_y(0,0)=$ .... (iii) $v_x(0,0)=$ .... (iv) $v_y(0,0)=$ ....

Solution: Note that $f(s,0)=\dfrac{(1+\frac{i}{2})s^5}{|s|^4}$, $f(0,t)=\dfrac{(1+\frac{i}{2})i\,t^5}{|t|^4}$

$\Rightarrow u(s,0)=\dfrac{s^5}{|s|^4}, v(s,0)=\dfrac{s^5}{2|s|^4}, u(0,t)=-\dfrac{t^5}{2|t|^4}, v(0,t)=\dfrac{t^5}{|t|^4}$

$\Rightarrow u_x(0,0)=1,\; v_x(0,0)=\dfrac{1}{2},\; u_y(0,0)=-\dfrac{1}{2},\; v_y(0,0)=1$

(5 mark for all correct, deduct 2 marks for each incorrect, no negative marks)

6. The image in $w-plane$ of rectangle $\{z=x+iy:-1<x<1,\;0<y<\pi\}$, under the mapping $w=e^z$, is the set

$\{w:$ ....$\}$

Solution: $w=u+iv;\; u=e^x\cos y,\; v=e^x\sin y$ imply $|w|=e^x$ so that the desired set is $\{w:\dfrac{1}{e}<|w|<e,\;\operatorname{Im}w>0\}$.

(5 marks, deduct 3 marks if *any* of the two conditions on w is not included in the answer)

7. For the harmonic function $v(x, y) = 2xy - e^{-2x} \sin 2y$, let $h(z)$ be a function analytic in the whole complex plane, such that $\operatorname{Im} h(z) = v(x, y)$. Then, $h(z)$, expressed as a function of z alone (not as a function of x, y), is

$h(z) =$ ....

Solution: $u_x = v_y \Rightarrow u_x = 2x - e^{-2x}\ 2\cos 2y \Rightarrow u = x^2 + e^{-2x} \cos 2y + g(y)$

$\Rightarrow u_y = -e^{-2x}\ 2\sin 2y + g'(y)$.

Now, $-v_x = u_y \Rightarrow -(2y + e^{-2x}\ 2\sin 2y) = -e^{-2x}\ 2\sin 2y + g'(y) \Rightarrow g(y) = -y^2 \Rightarrow u = x^2 + e^{-2x} \cos 2y - y^2$.

Therefore,

$$f(z) = x^2 + e^{-2x} \cos 2y - y^2 + i\ (2xy - e^{-2x} \sin 2y)$$

$$= z^2 + e^{-2z}$$

(5 marks, only 2 mark if answer is correctly expressed in terms of x, y)

8. The radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{2^{n^2}}{(2n)!} z^{n^2}$ is

....

Solution:

$$R = \lim_{n\to\infty} \left| \frac{a_{n-1}}{a_n} \right|^{1/(\lambda_n - \lambda_{n-1})} = \lim_{n\to\infty} \left| \frac{2^{(n-1)^2}}{(2n-2)!} \frac{(2n)!}{2^{n^2}} \right|^{1/\left(n^2-(n-1)^2\right)}$$

$$\lim_{n\to\infty} \left| \frac{2n(2n-1)}{2^{2n-1}} \right|^{1/(2n-1)} = \frac{1}{2}.$$

(5 marks)