Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MSO202A/MSO202 Assignment 3 Solutions
Introduction To Complex Analysis

The problems marked **(T)** *need an explicit discussion in the tutorial class. Other problems are for enhanced practice.*

1. (a) $\int_C |z|\bar{z}\ dz$, where C is the counterclockwise oriented semicircular part of the circle $|z| = 2$ lying in the second and third quadrants.

(T)(b) $\int_C |z|\frac{z}{\bar{z}}\ dz$, where C is the clockwise oriented boundary of the part of the annulus $2 \le |z| \le 4$ lying in the third and fourth quadrants.

(c) $\int_C (z-2a)^n\ dz$, where C is the semicircle $|z-2a| = R,\ 0 \le \arg(z-2a) \le \pi$.

Solution:

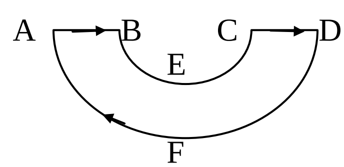
(a) A parametric equation of C is $z = 2e^{i\theta}$, $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Therefore the given integral is

$$\int_C |z|\bar{z}\ dz = 8i\int_{\pi/2}^{3\pi/2} d\theta = 8\pi i\ .$$

(b) $$\int_C |z|\frac{z}{\bar{z}}\ dz = \int_{AB} |z|\frac{z}{\bar{z}}\ dz + \int_{BEC} |z|\frac{z}{\bar{z}}\ dz + \int_{CD} |z|\frac{z}{\bar{z}}\ dz + \int_{DFA} |z|\frac{z}{\bar{z}}\ dz$$

$$= \int_{-4}^{-2} -x\ dx + \int_{\pi}^{2\pi} 2\frac{2e^{i\theta}}{2e^{-i\theta}}2ie^{i\theta}\ d\theta + \int_{2}^{4} x\ dx + \int_{0}^{-\pi} 4\frac{4e^{i\theta}}{4e^{-i\theta}}4ie^{i\theta}\ d\theta$$

$$= 6 + \frac{8}{3} + 6 - \frac{32}{3} = 4$$



(c) $$\int_C (z-2a)^n\ dz = \int_0^{\pi} R^n e^{in\theta} iR\, e^{i\theta}\ d\theta = iR^{n+1}\int_0^{\pi} e^{i(n+1)\theta}\ d\theta \qquad (i)$$

$$= \frac{R^{n+1}}{n+1}[(-1)^{n+1} - 1],\ for\ n+1 \ne 0.$$

The right hand integral in (i) is obviously πi for n+1=0.

2. Evaluate the integral $\int_C \frac{1}{\sqrt{z}}\,dz$, in each of the following cases:

(a) C is the counterclockwise oriented semicircular part of the circle $|z|=1$ in the upper half plane and $\sqrt{z}$ is defined so that $\sqrt{1}=1$.

(T)(b) C is the counterclockwise oriented semicircular part of the circle $|z|=1$ in the lower half plane and $\sqrt{z}$ is defined so that $\sqrt{1}=-1$.

(c) C is the clockwise oriented circle $|z|=1$ and $\sqrt{z}$ is defined so that $\sqrt{1}=1$.

(d) C is the counterclockwise oriented circle $|z|=1$ and $\sqrt{z}$ is defined so that $\sqrt{-1}=i$.

Solution:

(a) The two distinct values of $\sqrt{z}$, $z=r\,e^{i\theta}$, are given by $\sqrt{z}=\exp(\frac{\ln|z|}{2}+i\frac{(\theta+2n\pi)}{2}),\ n=0,1.$

$\sqrt{1}=1\Rightarrow n=0$. The parametric equation of C is $z=e^{it}, 0<t<\pi$

$$\Rightarrow \int_C \frac{1}{\sqrt{z}}dz=\int_0^{\pi}\frac{1}{e^{i(t/2)}}\,i\,e^{it}\ dt=i\frac{[e^{it/2}]_0^{\pi}}{i/2}=2(i-1).$$

(b) $\sqrt{1}=-1\Rightarrow n=1$. The parametric equation of C is $z=e^{it}, \pi<t<2\pi$

$$\Rightarrow \int_C \frac{1}{\sqrt{z}}dz=\int_{\pi}^{2\pi}\frac{1}{-e^{i(t/2)}}\,i\,e^{it}\ dt=-i\frac{[e^{it/2}]_{\pi}^{2\pi}}{i/2}=2(1+i).$$

(c) $\sqrt{1}=1\Rightarrow n=0$. The parametric equation of C is $z=e^{-it}, -2\pi<t<0$.

$$\Rightarrow \int_C \frac{1}{\sqrt{z}}dz=-\int_{-2\pi}^{0}\frac{1}{e^{-i(t/2)}}\,i\,e^{-it}\ dt=-i\frac{[e^{-it/2}]_{-2\pi}^{0}}{-(i/2)}=4.$$

(d) $\sqrt{-1}=i\Rightarrow n=0$. The parametric equation of C is $z=e^{it}, 0<t<2\pi$.

$$\Rightarrow \int_C \frac{1}{\sqrt{z}}dz=\int_0^{2\pi}\frac{1}{e^{i(t/2)}}\,i\,e^{it}\ dt=i\frac{[e^{it/2}]_0^{2\pi}}{(i/2)}=-4.$$

3. Without actually evaluating the integral, prove that

(T)(a) $\left|\int_C \frac{1}{z^2+1}dz\right|\le\frac{\pi}{3}$, where C is the arc of the circle $|z|=2$ from z = 2 to z = 2i lying in the first quadrant.

(b) $\left|\int_C (z^2-1)\,dz\right|\le \pi R(R^2+1)$, where C is the semicircle of radius R > 1 with center at the origin.

Solution:

(a) Length of $C=\pi$, *and on* C, $\left|\frac{1}{z^2+1}\right|\le\frac{1}{2^2-1}\Rightarrow\left|\int_C\frac{1}{z^2+1}dz\right|\le\frac{1}{2^2-1}\pi=\frac{\pi}{3}$. (by ML-Estimate)

(b) *Length of C =* πR, *and on C,* $|z^2-1|\le R^2+1$. Consequently, the desired inequality follows by ML-estimate

4. **(T)**Does Cauchy Theorem hold separately for the real or imaginary part of an analytic function f(z)? Why or why not?

Solution: No, Cauchy Theorem need not hold separately for real or imaginary part of an analytic function. Consider, for example f(z) = z and C : $|z|=1$. Then,

$$\int_{|z|=1} \operatorname{Re} z \; dz = \int_0^{2\pi} \cos\theta \; i \; e^{i\theta} \; d\theta = i\pi$$

$$\int_{|z|=1} \operatorname{Im} z \; dz = \int_0^{2\pi} \sin\theta \; i \; e^{i\theta} \; d\theta = -\pi$$

5. About the point z = 0, determine the Taylor series for each of the following functions:
   (T) $(i)\ \sqrt{z+2i}$ $\qquad$ $(ii)\ Log(z^2-3z+2)$

Solution:

(i) $\sqrt{z+2i} = \sqrt{2i}\ (1+\frac{z}{2i})^{1/2}$

$$= \sqrt{2}\ (e^{i\pi/2})^{1/2}[1+\frac{1}{2}\left(\frac{z}{2i}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{\lfloor 2}\left(\frac{z}{2i}\right)^2$$

$$+ \dots + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\dots\left(\frac{1}{2}-n+1\right)}{\lfloor n}\left(\frac{z}{2i}\right)^n + \dots]$$

$$= (1+i)[1+\frac{1}{2}\left(\frac{z}{2i}\right)+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1.3.5.....(2n-3)}{2^n\ \lfloor n}\left(\frac{z}{2i}\right)^n.$$

(ii) Log (z^2 – 3z + 2) = Log (1 – z) + Log (2 – z) + 2 $m\pi i$, for some m.

= Log (1 – z) + Log 2 + log (1 – $\frac{z}{2}$) + 2 m πi, for some m

= Log 2 + 2 m πi - $\sum_{n=1}^{\infty}\frac{z^n}{n}$ - $\sum_{n=1}^{\infty}\frac{(z/2)^n}{n}$

Therefore the Taylor series of Log (z^2 – 3z + 2) is given by

log (z^2 – 3z + 2) = log 2 + 2 m πi - $\sum_{n=1}^{\infty}(1+\frac{1}{2^n})\frac{z^n}{n}$

6. About the indicated point z = z_0, determine the Taylor series and its region of convergence for each of the following functions. In each case, does the Taylor series necessarily sums up to the function at every point of its region of convergence?

   $(i)\ \frac{1}{1+z},\ z_0 = 1$ **(T)** $(ii)\ \cosh z,\ z_0 = \pi i$ **(T)** $(iii)\ Log\, z,\ z_0 = -1+i$

Solution:

(i) $\frac{1}{1+z} = \frac{1}{2}\left[1+\frac{z-1}{2}\right]^{-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n$. The region of convergence of its Taylor series is $|z-1| < 2$. The Taylor series sums up to $\frac{1}{1+z}$ at every point of $|z-1| < 2$, since $\frac{1}{1+z}$ is analytic at every point of $|z-1| < 2$.

(ii) cosh z = cosh (z - π i + π i) = cos (i(z - π i) - π) = cos (i(z - π i)).

Therefore, $\cosh z = \sum_{n=0}^{\infty}\frac{(z-\pi i)^{2n}}{\lfloor 2n}$. The region of convergence of Taylor series is whole complex plane. The Taylor series sums up to cosh z at every point of the complex plane, since cosh z is analytic at every point of the complex plane.

(iii) $\left(\frac{d^n}{dz^n}Log\ z\right)_{z=-1+i} = (-1)^{n-1}\frac{\lfloor n-1}{(-1+i)^n},\ for\ n = 1, 2, \ldots\ldots$

$$\Rightarrow Log\ z = Log\ (-1+i) - \sum_{n=1}^{\infty}\frac{(1+i)^n}{n\,2^n}(z+1-i)^n .$$

The region of convergence of the Taylor series on RHS is $|z+1-i| < \sqrt{2}$. However, the Taylor series does not sum up to Log z at every point of $|z+1-i| < \sqrt{2}$, since Log z is not analytic on negative real axis, a part of which is contained in the disk $|z+1-i| < \sqrt{2}$, while the sum function represented by the above Taylor series is analytic at each point of this disk.

7. **(T)**Evaluate the integral $\int_C \frac{dz}{z(z^2+1)}$, for all possible choices of the contour C that does not pass through any of the points z = 0, $\pm\, i$.

Solution: Let curve C be oriented counterclockwise in the following cases. For clockwise oriented C the value of the integral will be negative of the value obtained in these cases.

Case1 (C does not enclose any of the points 0, $\pm\, i$): In this case, I = $\int_C \frac{dz}{z(z^2+1)}$ = 0, by Cauchy Theorem.

Case2

When C encloses only the point 0,

I = $\int_C \frac{(1/\left(z^2+1\right))}{z}\, dz = 2\pi i$, by Cauchy Integral Formula.

Similarly, when C encloses only the point i,

I = $\int_C \frac{(1/z\left(z+i\right))}{z-i}\, dz = 2\pi i \times \frac{1}{i\times 2i} = -\pi i$

and when C encloses only the point -i,

I = $\int_C \frac{(1/z\left(z-i\right))}{z+i}\, dz = 2\pi i \times \frac{1}{-i\times -2i} = -\pi i$.

Case3: When C encloses only the points 0, -i,

$$I = \int_{C_1} \frac{(1/(z^2+1))}{z}\, dz + \int_{C_2} \frac{(1/z(z-i))}{z+i}\, dz, \textit{ where } C_1 \textit{ and } C_2 \textit{ are sufficiently small circles around } 0 \textit{ and } -i \textit{ respectively.}$$

$= 2\pi i \times 1 + 2\pi i \times \frac{1}{-i\times -2i} = \pi i$, by Cauchy Integral Formula.

Other cases, i.e. when C encloses only the points 0, i or –i, i are treated similarly.

Case 4 (C encloses all of the points 0, i, -i): In this case

$$I = \int_{C_1} \frac{(1/(z^2+1))}{z}\, dz + \int_{C_2} \frac{(1/z(z-i))}{z+i}\, dz + \int_{C_3} \frac{(1/z(z+i))}{z-i}\, dz,$$

where C_1, C_2 *and* C_3 *are sufficiently small circles around* $0, -i$ *and* i *respectively.*

$$= 2\pi i \times 1 + 2\pi i \times \frac{1}{-i\times -2i} + 2\pi i \times \frac{1}{i\times 2i} = 0.$$

8. **(T)** Use Cauchy Theorem for multiply connected domains and Cauchy Integral Formula to evaluate the integral

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz \text{, C: the circle } |z| = 3 \text{ oriented counterclockwise.}$$

Solution: Let C_1, C_2 be the counterclockwise oriented circles of sufficiently small radius centered at 1 and 2 respectively. The integrand is an analytic function in the region lying between the circles C and C_1, C_2 Therefore, by Cauchy Theorem for multiply connected domains,

$$\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz = \int_{C_1} \frac{(\cos \pi z^2/(z-2))}{(z-1)}\, dz + \int_{C_2} \frac{(\cos \pi z^2/(z-1))}{(z-2)}\, dz$$

$$= 2\pi i\left(\frac{\cos \pi z^2}{z-2}\right)_{z=1} + 2\pi i\left(\frac{\cos \pi z^2}{z-1}\right)_{z=2} = 4\pi i.$$ (by Cauchy Integral Formula)

9. Evaluate

(T)(a) $\int_C \frac{e^{2z}}{z(z+1)^4}\, dz$, C: the circle $|z| = 2$ oriented clockwise

(b) $\int_C \frac{\sin z}{(z+\pi)^{2n}}\, dz$, C: the circle $|z+\pi| = 1$ oriented counterclockwise

Solution:

(a) Using Cauchy Theorem for multiply connected domains,

$$\textit{Given Integral} = \int_{|z|=\varepsilon_1} \frac{(e^{2z}/(z+1)^4)}{z}\, dz + \int_{|z+1|=\varepsilon_2} \frac{(e^{2z}/z)}{(z+1)^4}\, dz,$$

where $|z| = \varepsilon_1, |z+1| = \varepsilon_2$ *are sufficiently small clockwise oriented circles.*

$$= -\left[2\pi i\left(\frac{e^{2z}}{(z+1)^4}\right)_{z=0} + \frac{2\pi i}{\lfloor 3}\left\{\frac{d^3}{dz^3}\left(\frac{e^{2z}}{z}\right)\right\}_{z=-1}\right]$$ (by Cauchy Integral Formula for nth derivatives)

(b) Given Integral = $-\frac{2\pi i}{\lfloor 2n-1}\left(\frac{d^{2n-1}}{dz^{2n-1}}\sin z\right)_{z=-\pi} = \frac{2\pi i}{\lfloor 2n-1}(-1)^n$

10. If u is a harmonic function in $|z| < R$ *and* $0 < r < R$, show that

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta.$$

Solution: Let v be the harmonic conjugate of u in $|z| < R$ so that f(z) = u + i v is analytic in $|z| < R$. By Cauchy Integral Formula

$$f(0) = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z}\, dz = \frac{1}{2\pi i}\int_0^{2\pi} f(re^{i\theta})\; i\; d\theta.$$

Now taking the real part on both the sides gives the desired result.