Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MSO202A/MSO202 Assignment 4 Solutions
Introduction To Complex Analysis

The problems marked **(T)** *need an explicit discussion in the tutorial class. Other problems are for enhanced practice.*

1. **(T)** Give examples for the following:

(a)The radius of convergence of Taylor series of a function with center as some point a in the domain of analyticity D of the function is larger than the largest disk $|z-a|<R$ contained in D

(b) Two Taylor series with different centers represent the same analytic function in the intersection of their disks of convergence.

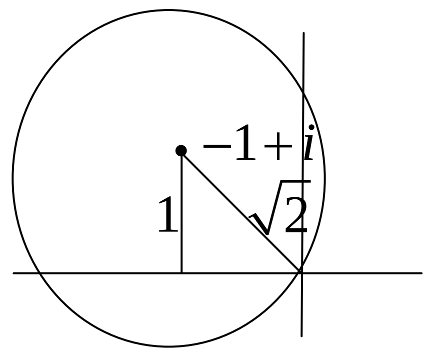
(c) The disk of convergence of Taylor series of a function is strictly contained in the domain of analyticity of a function.

Solution:

(a)The Taylor series of

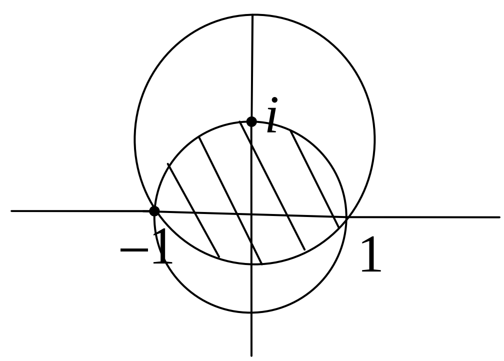
$Log\, z, -\pi < Arg\, z < \pi$, *centred at* $a=-1+i$, is

$$Log\, z = Log(-1+i) + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!(-1+i)^n}(z+1-i)^n \qquad (*)$$



which has radius of convergence $\sqrt{2}$, while the largest disk centered at $-1+i$ and contained in the domain of anlyticity of $Log\, z$ is $|z+1-i|<1$.

(b) The power series $\frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n$ around the point i has the radius of convergence $\sqrt{2}$ and the power series $\sum_{n=0}^{\infty} z^n$ has radius of convergence 1. On $D=\{|z-i|<\sqrt{2}\}\cap\{|z|<1\}\neq\phi$ both the series are Taylor series of the same function $\frac{1}{1-z}$.



(c) The function $\frac{1}{1-z}$ is analytic in the set $\boldsymbol{C}-\{1\}$ but its Taylor series $\sum_{n=0}^{\infty} z^n$ around $z=0$ has its disk of convergence $|z|<1$, strictly contained in $\boldsymbol{C}-\{1\}$.

2. Evaluate the following integrals on the indicated curves, all of them being assumed to be oriented in the counterclockwise direction:

   (T)(a) $\int_C \frac{1}{z^4 - 1}\, dz,\ \ C: |z| = 2$ (b) $\int_C \frac{2z^3 + z^2 + 4}{z^4 + 4z^2}\, dz,\ \ C: |z-2| = 4.$

Solution:

(a) Given Integral = $\frac{1}{4}\int_C \left(\frac{1}{z-1} - \frac{1}{z+1} + \frac{i}{z-i} - \frac{i}{z+i}\right) dz = 0$

(b) Given Integral = $\int_C \left(\frac{1}{z^2} + \frac{1}{z+2i} + \frac{1}{z-2i}\right) dz = 4\pi i$.

3. Evaluate the following integrals on the square C, oriented in the counterclockwise direction and having sides along the lines $x = \pm 2$ *and* $y = \pm 2$:

   (T)(i) $\int_C \frac{\cos z}{z(z^2+8)}\, dz$ **(T)**(ii) $\int_C \frac{\cosh z}{z^4}\, dz$.

Solution:

(i) *Given Integral* $= \int_C \frac{\cos z/(z^2+8)}{z}\, dz = 2\pi i \left(\frac{\cos z}{z^2+8}\right)_{z=0} = \frac{\pi i}{4}$,

since $z = \pm 2\sqrt{2}i$ *does not lie in the region bounded by* C.

(ii) Cosh z is analytic inside and on C, therefore

Given Integral = $\frac{2\pi i}{\lfloor 3}\left(\frac{d^3}{dz^3}\cosh z\right)_{z=0} = 0$

4. Using Liovuille Theorem, show that the functions exp(z), sin z, cos z , sinh z, cosh z are not bounded in the complex plane ***C***.

Solution: All the functions are entire. Had these functions been bounded in C, each would be a constant function (by Liouville Theorem), which they are not.

5. Show that every polynomial P(z) of degree n has exactly n zeros in the complex plane.

Solution: Let $P_n(z)$ be a polynomial of degree $n \geq 1$. and assume that it has no zeros in the complex plane ***C***. Then, the function $\varphi(z) = \frac{1}{P_n(z)}$ (i) is an entire function (ii) is bounded in ***C*** (*since* $P_n(z) \to \infty$ as $z \to \infty$)

Therefore, by Liouville's Theorem, $\varphi(z)$ is constant. $\Rightarrow$ $P_n(z)$ is also a constant function, a contradiction. Thus, $P_n(z)$ has at least one zero, say a_1 of multiplicity m_1. If $m_1 = n$, the desired result follows.

If $m_1 \neq n$, the polynomial $\frac{P_n(z)}{(z-a_1)^{m_1}}$, is a non-constant polynomial of degree $n - m_1$ and a repetition of the above arguments gives that it has at least one zero, say a_2 of multiplicity m_2.

The above process continues till $m_1+m_2+...+m_k=n$ for some natural number $k\geq 1$. It therefore follows that $P_n(z)$ has zeros at $a_1,a_2,...,a_k$ of respective multiplicities $m_1,m_2...,m_k$ such that $m_1+m_2+...+m_k=n$.

6. If f is an entire function and $|f(z)|\leq MR^{n_0}$ in $|z|\leq R$, prove that f is a polynomial of degree at most n_0.

Solution: By Taylor's Theorem, expand $f(z)=\sum_{n=0}^{\infty}a_nz^n$ in $|z|<R_0$. The same expansion is valid for all $R>R_0$. By Cauchy Estimate, $\left|f^{(n)}(0)\right|\leq\dfrac{n!M(R)}{R^n}$, where $M(R)=\max_{|z|=R}|f(z)|$

$\therefore |a_n|\leq\dfrac{MR^{n_0}}{R^n}=MR^{n_0-n}\to 0$ *as* $n\to\infty$, *if* $n>n_0$. $\Rightarrow f$ is a polynomial of degree at most n_0.

7. Let $f(z)$ be analytic in $|z|\leq R$. Prove that, for 0 < r < R,

$$f(re^{i\varphi})=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\varphi)}f(\mathrm{Re}^{i\varphi})\,d\varphi \text{ (called Poisson Integral Formula).}$$

Solution: Let $|a|<R$. By Cauchy Integral Formula, $f(a)=\dfrac{1}{2\pi i}\displaystyle\int_{|z|=R}\frac{f(z)}{z-a}\,dz$ ....(i).

Since the point $\dfrac{R^2}{\bar{a}}$ lies outside the circle $|z|=R$, by Cauchy Theorem, $0=\dfrac{1}{2\pi i}\displaystyle\int_{|z|=R}\frac{f(z)}{z-(R^2/\bar{a})}\,dz$ ....(ii).

Adding (i) and (ii), $f(a)=\dfrac{1}{2\pi i}\displaystyle\int_{|z|=R}\frac{f(z)(R^2-a\bar{a})}{(z-a)(R^2-\bar{a}z)}\,dz$. Now, letting $a=re^{i\theta}$ *and* $z=Re^{i\varphi}$ in the above equation, gives

$$f(re^{i\theta})=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\varphi})(R^2-r^2)}{Re^{i\varphi}(1-\frac{a}{R}e^{-i\varphi})(R^2-\bar{a}Re^{i\varphi})}iRe^{i\varphi}\,d\varphi=\frac{1}{2\pi}\int_{|z|=R}\frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\varphi)}f(\mathrm{Re}^{i\varphi})\,d\varphi.$$

8. **(T)** Evaluate $\displaystyle\int_\Gamma\frac{1}{z^4}dz$, where Γ is the part of clockwise oriented ellipse $\dfrac{(x-3)^2}{1}+\dfrac{y^2}{4}=1$ lying in the upper half-plane $\{z:\operatorname{Im}z>0\}$.

Solution: Let Γ^* be the clockwise oriented closed curve consisting of the part of given ellipse in upper half-plane and the line segment L with initial point $(4,0)$ and end point $(2,0)$. Since the function $1/z^4$ is analytic inside and on Γ^*, $\displaystyle\int_\Gamma\frac{1}{z^4}dz=\int_{-L}\frac{1}{z^4}dz=\int_2^4\frac{1}{x^4}dx=-\frac{1}{3}(x^{-3})_2^4=\frac{7}{192}$.

9. Find the order of the zero z = 0 for the following functions:

$(i)\ z^2(e^{z^2}-1)$ **(T)** $(ii)\ 6\sin z^3 + z^3(z^6-6)$ **(T)**(iii) $e^{\sin z} - e^{\tan z}$

Solution:

(i) The first nonzero term in the Taylor series of the given function around z = 0, contains z^4, therefore its zero at z = 0 is of order 4

(ii) The first nonzero term in the Taylor series of the given function around z = 0, contains z^{15} therefore its zero at z = 0 is of order 15.

(iii) The first nonzero term in the Taylor series of the given function around z = 0, contains z^3 therefore its zero at z = 0 is of order 3.

10. Find the order of all the zeros of the following functions:

$(i)\ z\sin z$ **(T)** $(ii)(1-e^z)(z^2-4)^3$ **(T)** $(iii)\ \dfrac{\sin^3 z}{z}$

Solution:

(i) zero of order 2 at z = 0, simple zeros at z = $n\pi$, n = nonzero integer.

(ii) zero of order 3 at z = ± 2, simple zeros at z = $2n\pi i$, n = nonzero integer.

(iii) zero of order 2 at z = 0, zeros of order 3 at z = $n\pi$, n = nonzero integer.

11. **(T)**Does there exist a function f(z) ($not\ identically\ zero$) that is analytic in $|z|<1$ and has zeros at the following indicated set of points ? Why or why not?

(i) $S_1 = \{\frac{1}{n} : n\ is\ a\ natural\ number\}$ (ii) $S_2 = \{1-\frac{1}{n} : n\ is\ a\ natural\ number\}$

(iii) $S_3 = \{z : |z|<1, \mathrm{Re}(z)=0\}$ $(iv)\ S_4 = \{z = \frac{1}{2} + iy : -\frac{1}{2} < y < \frac{1}{2}\}$.

Solution:

(i) No, since limit point of S_1 is 0 which lies in $|z|<1$, so 0 would be a non-isolated zero of f(z) (ii) Yes, since limit point of S_2 does not lie in $|z|<1$ (iii) No, since limit points of S_3 lie in $|z|<1$ (iv) No, since limit points of S_4 lie in $|z|<1$.

G.P.Kapoor