Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MSO202A/MSO202 Assignment 5 Solutions
Introduction To Complex Analysis

The problems marked **(T)** *need an explicit discussion in the tutorial class. Other problems are for enhanced practice.*

1. Expand each of the following functions in Laurent series in the neighbourhood of the indicated points z_0 and, in each case, determine the largest domain where the resulting Laurent series converges:
   (i) $\frac{1}{z(1-z)}$, $z_0 = 0, 1$ and ∞ **(T)** (ii) $z^2 e^{1/z}$, $z_0 = 0$ and ∞ (iii) $\frac{1}{z^2+1}$, $z_0 = -i, \infty$.

Solution:

(i) *For $z_0 = 0$*: $\frac{1}{z(1-z)} = \frac{1}{z}(1-z)^{-1} = \frac{1}{z} + \sum_{n=0}^{\infty} z^n$ is the desired Laurent series. The largest annulus of convergence for this Laurent series is $0 < |z| < 1$.

For $z_0=1$: $\frac{1}{z(1-z)} = \frac{1}{1-z}[1-(1-z)]^{-1} = \frac{1}{1-z} + \sum_{n=0}^{\infty} (1-z)^n$ is the desired Laurent series. The largest annulus of convergence for this Laurent series is $0 < |z-1| < 1$.

For $z_0 = \infty$: $\frac{1}{z(1-z)} = -\frac{1}{z^2}(1-\frac{1}{z})^{-1} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+2}}$ is the desired Laurent series. The largest annulus of convergence for this Laurent series is $1 < |z| < \infty$.

(ii) *For $z_0 = 0$*: f(z) = $z^2 e^{1/z} = z^2 \sum_{n=0}^{\infty} \frac{1}{\lfloor n} \frac{1}{z^n} = z^2 + z + \frac{1}{2} + \sum_{n=3}^{\infty} \frac{1}{\lfloor n} \frac{1}{z^{n-2}}$ is the desired Laurent series. The largest annulus of convergence for this Laurent series is $0 < |z| < \infty$.

For $z_0 = \infty$: Expand f(1/z) in the neighbourhood of $z_0 = 0$ and replace z by 1/z in the resulting Laurent series. Observe that the Laurent series of f(z) in the neighbourhood of $z_0 = \infty$ is the same as its Laurent series in the neighbourhood of $z_0 = 0$. Both have the same annulus of convergence $0 < |z| < \infty$.

(iii) *For $z_0 = -i$*: $\frac{1}{z^2+1} = \frac{1}{-2i\,(z+i)}[1 - \frac{z+i}{2i}]^{-1} = \frac{i}{2(z+i)} + \frac{i}{2}\sum_{n=1}^{\infty} \frac{(z+i)^{n-1}}{(2i)^n}$
is the desired Laurent series. The largest annulus of convergence for this Laurent series is $0 < \left|\frac{z+i}{2i}\right| < 1$, *or* $0 < |z+i| < 2$.

For $z_0 = \infty$: $\frac{1}{z^2+1} = \frac{i}{2}\left[\frac{1}{(z+i)} - \frac{1}{(z-i)}\right] = \frac{i}{2z}\sum_{n=0}^{\infty} (-1)^n \frac{i^n}{z^n} - \frac{i}{2z}\sum_{n=0}^{\infty} \frac{i^n}{z^n}$
is the desired Laurent series. The largest annulus of convergence for this Laurent series is $\left|\frac{i}{z}\right| < 1$, *or* $1 < |z| < \infty$.

2. For each of the following functions, determine the nature of its isolated singularities by considering the relevant Laurent series

(T) $(i)\ \dfrac{1-\cosh z}{z^3}$ $\quad (ii)\ \dfrac{\sin z}{z}$ $\quad (iii)\ e^z$ $\quad$ (iv) $1 + 2z + 7z^3 + 3z^7$

Solution:

(i) Around z = 0, Laurent series of $\dfrac{1-\cosh z}{z^3} = -\dfrac{1}{\underline{|2}\,z} - \dfrac{z}{\underline{|4}} - \ \ldots\ldots \Rightarrow \textit{simple pole at } z = 0$

For nature of singularity at z = ∞, Laurent series of $\dfrac{1-\cosh(1/z)}{(1/z)^3}$ around z = 0

$= z^3[-\dfrac{1}{\underline{|2}\,z^2} - \dfrac{1}{\underline{|4}\,z^4} - \dfrac{1}{\underline{|6}\,z^6}\ldots\ldots] \Rightarrow$ z = 0 is an essential singularity of $\dfrac{1-\cosh(1/z)}{(1/z)^3}$

$\Rightarrow$ z = ∞ is an essential singularity of $\dfrac{1-\cosh z}{z^3}$.

(ii) Around z = 0, Laurent series of $\dfrac{\sin z}{z} = 1 - \dfrac{z^2}{\underline{|3}} + \dfrac{z^4}{\underline{|5}} - \ldots. \Rightarrow z = 0\ \textit{ is a removable singularity}$

For nature of singularity at z = ∞, Laurent series of $z \sin\dfrac{1}{z}$ around z = 0

$= 1 - \dfrac{1}{\underline{|3}\,z^2} + \dfrac{1}{\underline{|5}\,z^5} - \ldots\ldots$

$\Rightarrow$ z = 0 is an essential singularity of $z\sin\dfrac{1}{z}$ $\Rightarrow$ z = ∞ is an essential singularity of $\dfrac{\sin z}{z}$.

(iii) For nature of singularity at z = ∞, Laurent series of $e^{1/z}$ around z = 0

$= 1 + \dfrac{1}{z} + \dfrac{1}{\underline{|2}\,z^2}\ldots\ldots$

$\Rightarrow$ z = 0 is an essential singularity of $e^{1/z}$ $\Rightarrow$ z = ∞ is an essential singularity of e^z .

(iv) For nature of singularity at z = ∞, observe that Laurent series of $1 + \dfrac{2}{z} + \dfrac{7}{z^3} + \dfrac{3}{z^7}$ around z = 0 is this expression itself $\Rightarrow$ z = 0 is a pole of order 7 of $1 + \dfrac{2}{z} + \dfrac{7}{z^3} + \dfrac{3}{z^7}$.

$\Rightarrow$ z = ∞ is a pole of order 7 of $1 + 2z + 7z^3 + 3z^7$.

3. For the following functions, determine the residues at each of their isolated singularities in the extended complex plane:

$(i)\ \dfrac{1}{z^3 - z^5}$ $\quad (ii)\ \dfrac{z^{2n}}{(1+z)^n}$ $\quad$ **(T)** $(iii)\ e^z\, e^{1/z}$

Solution:

(i) $\operatorname*{Res}_{z=1} \dfrac{1}{z^3 - z^5} = \lim_{z\to 1}(z-1)\dfrac{1}{z^3 - z^5} = -\dfrac{1}{2}$, $\operatorname*{Res}_{z=-1} \dfrac{1}{z^3 - z^5} = \lim_{z\to -1}(z+1)\dfrac{1}{z^3 - z^5} = -\dfrac{1}{2}$

$\underset{z=0}{\text{Res}}\frac{1}{z^3-z^5}$ = coefficient of (1/z) in the Laurent's expansion of the function around z = 0

= coefficient of (1/z) in $\frac{1}{z^3}(1+z^2+z^4+.....)$ = 1.

$\underset{z=\infty}{\text{Res}}\frac{1}{z^3-z^5}=-(1-\frac{1}{2}-\frac{1}{2})=0.$

$(ii)\ \underset{z=-1}{\text{Res}}\frac{z^{2n}}{(1+z)^n}=\lim_{z\to-1}\frac{1}{\lfloor n-1}\frac{1}{\lfloor n-1}\frac{d^{n-1}}{dz^{n-1}}\left[(1+z)^n\frac{z^{2n}}{(1+z)^n}\right]$

$=\frac{1}{\lfloor n-1}(2n)(2n-1)....(2n-(n-2))\lim_{z\to-1}z^{n+1}$

$=(-1)^{n+1}\frac{\lfloor 2n}{\lfloor n-1\,\lfloor n+1}$ = p(n) (say). $\Rightarrow\underset{z=\infty}{\text{Res}}\frac{z^{2n}}{(1+z)^n}=-p(n)$.

$(iii)\ \underset{z=0}{\text{Res}}\left[e^z\ e^{1/z}\right]=$ coefficient of (1/z) in the Laurent's expansion of $e^z e^{1/z}$ around z = 0

= coefficient of (1/z) in $(1+z+\frac{z^2}{\lfloor 2}+....+\frac{z^n}{\lfloor n}+...)(1+\frac{1}{z}+\frac{1}{\lfloor 2\,z^2}+....+\frac{1}{\lfloor n\,z^n}+...)$

$=\sum_{n=0}^{\infty}\frac{1}{\lfloor n\,\lfloor n+1}$ = $q(n)$ (say) $\Rightarrow\underset{z=\infty}{\text{Res}}[e^z e^{1/z}]=-q(n)$

4. Find residues of the following functions at all its poles:

(i) cot z $\qquad(ii)\ \frac{z}{z^n-1}\qquad$ **(T)** $(iii)\frac{z(z^3+5)}{(z-1)^3}$

Solution:

(i) The poles of cot z are at z = nπ, $n=0,\pm1,\pm2,\ldots$, all of which are simple $\Rightarrow\underset{z=n\pi}{\text{Res}}\cot z=\frac{\cos n\pi}{\cos n\pi}=1$, for all n.

(ii) The poles of $\frac{z}{z^n-1}$ are at $z_k=e^{2\pi ik/n}$, k = 0, 1, ..., (n – 1), all of which are simple $\Rightarrow$

$\underset{z=z_k}{\text{Res}}\frac{z}{z^n-1}=\left[\frac{z}{nz^{n-1}}\right]_{z=z_k}=\frac{z_k^2}{n}=\frac{1}{n}e^{4\pi ik/n}$.

(iii) The poles of $\frac{z(z^3+5)}{(z-1)^3}$ are at z = 1 (pole of order 3) and ∞ (simple).

$\underset{z=1}{\text{Res}}\frac{z(z^3+5)}{(z-1)^3}=\frac{1}{\lfloor 2}\left[\frac{d^2}{dz^2}\left(z(z^3+5)\right)\right]_{z=1}=\frac{1}{\lfloor 2}\times12=6.$

and

$\underset{z=\infty}{\text{Res}}\frac{z(z^3+5)}{(z-1)^3}=-6$.

5. Evaluate

(i) $\int\limits_{|z|=2} \tan z \, dz$ (ii) $\int\limits_{|z|=2} \frac{1}{\sin 2z} dz$ **(T)** (iii) $\int\limits_{|z|=8} \frac{e^{z/3}}{\sinh z} dz$

The integration in each case being in anticlockwise direction.

Solution:

(i) By Cauchy Residue Theorem, $\int\limits_{|z|=2} \tan z \, dz = 2\pi i[\operatorname*{Res}_{z=\pi/2} \tan z + \operatorname*{Res}_{z=-\pi/2} \tan z] = -4\pi i$

(ii) By Cauchy Residue Theorem,

$$\int\limits_{|z|=2} \frac{1}{\sin 2z} dz = 2\pi i[\operatorname*{Res}_{z=0} \frac{1}{\sin 2z} + \operatorname*{Res}_{z=\pi/2} \frac{1}{\sin 2z} + \operatorname*{Res}_{z=-\pi/2} \frac{1}{\sin 2z}] = 2\pi i[\frac{1}{2} - \frac{1}{2} - \frac{1}{2}] = -\pi i$$

(iii) By Cauchy Residue Theorem,

$$\int\limits_{|z|=8} \frac{e^{z/3}}{\sinh z} dz = 2\pi i[\operatorname*{Res}_{z=0} \frac{e^{z/3}}{\sinh z} + \operatorname*{Res}_{z=\pi i} \frac{e^{z/3}}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \frac{e^{z/3}}{\sinh z} + \operatorname*{Res}_{z=2\pi i} \frac{e^{z/3}}{\sinh z} + \operatorname*{Res}_{z=-2\pi i} \frac{e^{z/3}}{\sinh z}$$

$$= 2\pi i[1 + (-e^{i\pi/3}) + (-e^{-i\pi/3}) + e^{2i\pi/3} + e^{-2i\pi/3}] = 1 - 2\cos\frac{\pi}{3} + 2\cos\frac{2\pi}{3}$$

6. **(T)** Show that the functions that are analytic in the whole complex plane and have a non-essential isolated singularity at ∞ are polynomials.

Solution: Let f(z) be analytic in whole complex plane and has a nonessential singularity at ∞. Let the Taylor expansion of f(z) around 0 be $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then, $f(1/z) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n$ will have only finitely many nonzero terms (since, at z = 0, f(1/z) has a nonessential singularity, i.e. either a pole or a removable singularity) $\Rightarrow a_n = 0$ *for all* $n > m \Rightarrow$ f(z) is a polynomial.

7. Evaluate the following integrals using Cauchy Residue Theorem:

(i) $\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta} d\theta$ **(T)**(ii) $\int_0^{2\pi} \cos^{2n}\theta \, d\theta$

Solution:

(i) Putting $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \cos\theta = \frac{z+z^{-1}}{2}, \sin\theta = \frac{z-z^{-1}}{2i}$,

$$\int_0^{2\pi} \frac{1+\sin\theta}{3+\cos\theta} d\theta = -\int\limits_{|z|=1} \frac{z^2+2iz-1}{z(z^2+6z+1)} dz = -[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-3+2\sqrt{2}} f(z)] = -2\pi i[-1 + \frac{i}{2\sqrt{2}} + 1] = \frac{\pi}{\sqrt{2}}.$$

(ii) Putting $z = e^{i\theta}, d\theta = \frac{dz}{iz}, \cos\theta = \frac{z+z^{-1}}{2}$,

$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = -\frac{i}{2^{2n}} \int_{|z|=1} \frac{(z^2+1)^{2n}}{z^{2n+1}}$$

$$= -\frac{i}{2^{2n}} 2\pi i \operatorname{Res}_{z=0} \frac{(z^2+1)^{2n}}{z^{2n+1}} = \frac{2\pi}{2^{2n}} C_n^{2n} \ (\because \frac{(z^2+1)^{2n}}{z^{2n+1}} = \sum_{k=0}^{\infty} C_k^{2n} z^{2(n-k)-1})$$

8. Use Cauchy Residue Theorem to evaluate (i) $\int_0^{\infty} \frac{\cos x}{(x^2+1)^2} dx$ **(T)**(ii) $\int_0^{\infty} \frac{\sin^2 x}{1+x^2} dx$

Solution:

(i) $\int_0^{\infty} \frac{\cos x}{(x^2+1)^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx = \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2} dx = \frac{1}{2} Re\ I,\ (say)$

Let $f(z) = \frac{1}{(z^2+1)^2}$. Then, $|f(z)| \le \frac{1}{(R^2-1)^2}$ *on the semicircle* $C_R : |z| = R,\ \operatorname{Im}(z) > 0$.

$\Rightarrow$ *For* z *on* $C_R,\ f(z) \to 0$ *as* $R \to \infty \Rightarrow \int_{C_R} \frac{e^{iz}}{(z^2+1)^2} dz \to 0$ *as* $R \to \infty$..

$$\Rightarrow I = P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2} dx = 2\pi i \operatorname{Res}_{z=i} \frac{e^{iz}}{(z^2+1)^2} = 2\pi i \lim_{z\to i} \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right) = 2\pi i \frac{1}{2ei} = \frac{\pi}{e}.$$

$$\Rightarrow \int_0^{\infty} \frac{\cos x}{(x^2+1)^2} dx = \frac{\pi}{2e}$$

(ii) $\int_0^{\infty} \frac{\sin^2 x}{1+x^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin^2 x}{1+x^2} dx = \frac{1}{4} \int_{-\infty}^{\infty} \frac{1-\cos 2x}{1+x^2} dx = \frac{1}{4} I,\ (say)$

Now, I = $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx - \int_{-\infty}^{\infty} \frac{\cos 2x}{1+x^2} dx = I_1 - I_2,\ (say)$.

$I_1 = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = 2\pi i \ \operatorname{Res}_{z=i} \frac{1}{1+z^2} = 2\pi i \frac{1}{2i} = \pi$

(since the degree of the denominator is 2 greater than degree of the numerator in the integrand).

$I_2 = \int_{-\infty}^{\infty} \frac{\cos 2x}{1+x^2} dx = Re \int_{-\infty}^{\infty} \frac{e^{2ix}}{1+x^2} dx = Re\ (2\pi i (\operatorname{Res}_{z=i} \frac{e^{2iz}}{1+z^2})) = Re\ (2\pi i \frac{e^{-2}}{2i}) = \frac{\pi}{e^2}$

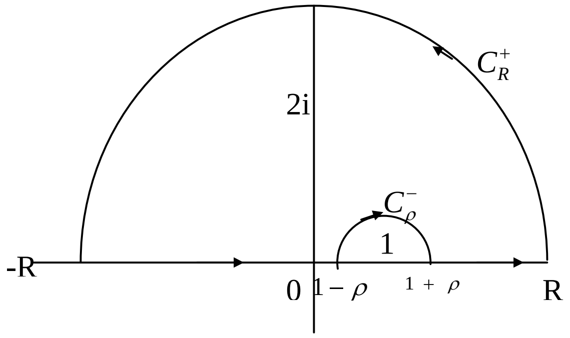
(since, *for* $|z| = R,\ \operatorname{Im}(z) > 0,\ \left|\frac{1}{1+z^2}\right| \le \frac{1}{R^2-1} \to 0$ *as* $R \to \infty$)

Therefore, $\int_0^{\infty} \frac{\sin^2 x}{1+x^2} dx = \frac{1}{4}(\pi - \frac{\pi}{e^2})$.

9. **(T)**Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x-1)(x^2+4)} dx$ by indenting the singularity on real axis.

Solution.

Let $f(z) = \frac{1}{(z-1)(z^2+4)}$ and $C = [-R, 1-\rho] \cup C_\rho^- \cup [1+\rho, R] \cup C_R^+$ be the simple closed curve as indicated in the figure.



$$\int_C \frac{1}{(z-1)(z^2+4)} dz$$

$$= \int_{[-R,1-\rho]} \frac{1}{(x-1)(x^2+4)} dx + \int_{C_\rho^-} \frac{1}{(z-1)(z^2+4)} dz$$

$$+ \int_{[1+\rho,R]} \frac{1}{(x-1)(x^2+4)} dx + \int_{C_R^+} \frac{1}{(z-1)(z^2+4)} dz \qquad (*)$$

On C_R^+, $\left| \int_{C_R^+} \frac{1}{(z-1)(z^2+4)} dz \right| \le \frac{\pi R}{(R-1)(R^2-4)} \to 0 \text{ as } R \to \infty$

Further, $\int_{C_\rho^-} \frac{1}{(z-1)(z^2+4)} dz = \int_{C_\rho^-} \frac{1}{5(z-1)} dz + \int_{C_\rho^-} g(z)\, dz$,

where g(z) is analytic at z = 1, and so is bounded by M (say) on C_ρ^- if ρ is sufficiently small.

Now, $\int_{C_\rho^-} \frac{1}{5(z-1)} dz = \int_{\pi}^{0} \frac{i\rho e^{i\theta}}{5i\rho e^{i\theta}} d\theta = \frac{-\pi i}{5}$. Further, $\left| \int_{C_\rho^-} g(z)\, dz \right| \le M\pi\rho \to 0 \text{ as } \rho \to 0$. Therefore,

$\int_{C_\rho^-} \frac{1}{(z-1)(z^2+4)} dz \to \frac{-\pi i}{5}$ as $\rho \to 0$. ***(Alternatively, put z – 1 = $\rho e^{i\theta}$ in the integral on C_ρ^-, and take limit $\rho \to 0$ inside the integral).*** Since,

$$\int_C \frac{1}{(z-1)(z^2+4)} dz = 2\pi i \operatorname{Res}_{z=2i} \frac{1}{(z-1)(z^2+4)} = \frac{\pi}{2(2i-1)}$$

making $\rho \to 0$ *and* $R \to \infty$ in (*) gives, P.V. $\int_{-\infty}^{\infty} \frac{1}{(x-1)(x^2+4)} dx = \frac{\pi}{2(2i-1)} + \frac{\pi i}{5}$.