SIAM J. NUMER. ANAL.
Vol. 44, No. 2, pp. 655–676

GENERALIZED CUBIC SPLINE FRACTAL INTERPOLATION FUNCTIONS*

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Abstract. We construct a generalized C^r-Fractal Interpolation Function (C^r-FIF) f by prescribing any combination of r values of the derivatives $f^{(k)}$, $k = 1, 2, \ldots, r$, at boundary points of the interval $I = [x_0, x_N]$. Our approach to construction settles several questions of Barnsley and Harrington [*J. Approx Theory*, 57 (1989), pp. 14–34] when construction is not restricted to prescribing the values of $f^{(k)}$ at only the initial endpoint of the interval I. In general, even in the case when r equations involving $f^{(k)}(x_0)$ and $f^{(k)}(x_N)$, $k = 1, 2, \ldots, r$, are prescribed, our method of construction of the C^r-FIF works equally well. In view of wide ranging applications of the classical cubic splines in several mathematical and engineering problems, the explicit construction of cubic spline FIF $f_\Delta(x)$ through *moments* is developed. It is shown that the sequence $\{f_{\Delta_k}(x)\}$ converges to the defining data function $\Phi(x)$ on two classes of sequences of meshes at least as rapidly as the square of the mesh norm $\|\Delta_k\|$ approaches to zero, provided that $\Phi^{(r)}(x)$ is continuous on I for $r = 2, 3$, or 4.

Key words. fractal, iterated function system, fractal interpolation function, spline, cubic spline fractal interpolation function, convergence

AMS subject classifications. 26A18, 37N30, 41A30, 65D05, 65D07, 65D10

DOI. 10.1137/040611070

1. Introduction.

With the advent of fractal geometry [2], the use of stochastic or deterministic fractal models [3, 4, 5] has significantly enhanced the understanding of complexities in nature and different scientific experiments. Hutchinson [6] has studied the deterministic fractal model based on the theory of Iterated Function System (IFS). Using IFS, Barnsley [3, 7] has introduced the concept of Fractal Interpolation Function (FIF) for approximation of naturally occurring functions showing some sort of self-similarity under magnification. A FIF is the fixed point of the Read–Bajraktarević operator acting on different function spaces. Generally, affine FIFs are nondifferentiable functions and the fractal dimensions of their graphs are nonintegers. The generation of FIF codes provides a powerful technique for compression of images, speeches, time series, and other data; see, e.g., [8, 9, 10].

If the experimental data are approximated by a C^r-FIF f, then one can use the fractal dimension of $f^{(r)}$ as a quantitative parameter for the analysis of experimental data. The differentiable C^r-FIF differs from the classical spline interpolation by a functional relation that gives self-similarity on small scales. Barnsley and Harrington [1] have introduced an algebraic method for the construction of a restricted class of C^r-FIF f, which interpolates the prescribed data by providing values of $f^{(k)}$, $k = 1, 2, \ldots, r$, at the initial endpoint of the interval. However, in their method of construction, specifying boundary conditions similar to those for classical splines has been found to be quite difficult to handle. Massopust [11] has attempted to generalize work in [1] by constructing smooth fractal surfaces via integration.

*Received by the editors July 5, 2004; accepted for publication (in revised form) November 22, 2005; published electronically March 17, 2006. This work was partially supported by the Council of Scientific and Industrial Research, India, grant 9/92(160)/98-EMR-I.
http://www.siam.org/journals/sinum/44-2/61107.html

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In the present paper, a method of construction of a C^r-fractal function is developed by removing the requirement of prescribing the values of integrals of the given FIF only at the initial endpoint x_0. Thus, a C^r-fractal function is constructed when successive r values of integrals of a FIF are prescribed in any combination at boundary points of the interval. Further, a general method is proposed to construct an interpolating C^r-FIF for the prescribed data with all possible boundary conditions. The complex algebraic method proposed in [1] uses complicated matrices and particular types of end conditions. Using the functional relations present between the values of the C^r-FIF that involve endpoints of the interval, our approach does not need the complex algebraic method in [1]. Our construction settles several queries of Barnsley and Harrington [1] such as (i) which boundary point conditions lead to uniqueness of a C^r-FIF, (ii) what happens if horizontal scalings are in reverse direction and (iii) how to build up the moment integrals theory in this case. The advantage of such a spline FIF construction is that, for prescribed data and given boundary conditions, one can have an infinite number of spline FIFs depending on the vertical scaling factors, giving thereby a large flexibility in the choice of differentiable C^r-FIFs according to the need of an experiment.

Due to the importance of the cubic splines in computer graphics, CAGD, FEM, differential equations, and several engineering applications [12, 13, 14, 15], cubic spline FIF $f_\Delta(x)$ on a mesh Δ is constructed through *moments* $M_n = f_\Delta^{''}(x_n), n = 0, 1, 2, \ldots, N$. These cubic spline FIFs may have any types of boundary conditions as in classical splines. It is shown that the sequence $\{f_{\Delta_k}(x)\}$ converges to the defining data function $\Phi(x)$ on two classes of sequences of meshes at least as rapidly as the square of the mesh norm $\|\Delta_k\|$ converging to zero, provided that $\Phi^{(r)}(x)$ is continuous on $[x_0, x_N]$ for $r = 2, 3$, or 4.

In section 2, some basic results for FIFs are given and a general method for construction of a C^r-FIF with different boundary conditions is enunciated after developing a basic calculus of C^1-FIFs. The construction of a generalized cubic spline FIF through moments is described in section 3 with all possible boundary conditions, as in the classical splines. In section 4, two classes of sequences of meshes are defined and the convergence of suitable sequence of cubic spline FIFs $\{f_{\Delta_k}\}$ to $\Phi \in C^r[x_0, x_N]$, $r = 2, 3$, or 4, is established. Finally, in section 5, the results obtained in section 3 are illustrated by generating certain examples of cubic spline FIFs for a given data and two different sets of vertical scaling factors.

2. A general method for construction of C^r-FIF.

We give the basics of the general theory of FIFs and develop the calculus of C^1-FIFs in section 2.1. The principle of construction of a C^r-FIF that interpolates the given data is described in section 2.2.

2.1. Preliminaries and calculus of C^1-FIFs.

Barnsley et al. [1, 3, 8, 16, 17] have developed the theory of FIF and its extensive applications. In the following, some of the notations and results of FIF theory, which we will later need, are described.

Let K be a complete metric space with metric d and $\mathcal{H}$ be the set of nonempty compact subsets of K. Then, $\{K; \omega_n, n = 1, 2, \ldots, N\}$ is an iterated function system (IFS) if $\omega_n : K \to K$ is continuous for $n = 1, 2, \ldots, N$. An IFS is called hyperbolic if $d(\omega_n(x), \omega_n(y)) \le sd(x, y)$ for all $x, y \in K, n = 1, 2, \ldots, N$ and $0 \le s < 1$. Set $W(A) = \bigcup_{n=1}^N \omega_n(A)$ for $A \in \mathcal{H}$. The following proposition gives a condition on an IFS to have a unique attractor.

PROPOSITION 2.1 (see [3]). *Let $\{K; \omega_n, n = 1, 2, \ldots, N\}$ be a hyperbolic IFS.*

Then, it has an unique attractor G such that $h(W^m(A), G) \to 0$ as $m \to \infty$, where $h(.,.)$ is the Hausdorff metric.

Suppose a set of data points $\{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, 2, \dots, N\}$ is given, where $x_0 < x_1 < \cdots < x_N$ and $I = [x_0, x_N]$. Set $K = I \times D$, where D is a suitable compact set in $\mathbb{R}$. Let $L_n : I \longrightarrow I_n = [x_{n-1}, x_n]$ be the affine map satisfying

$$L_n(x_0) = x_{n-1}, \quad L_n(x_N) = x_n \tag{2.1}$$

and $F_n : K \longrightarrow D$ be a continuous function such that

$$\left.\begin{aligned} F_n(x_0, y_0) = y_{n-1},\ F_n(x_N, y_N) = y_n \\ |F_n(x, y) - F_n(x, y^*)| \le \alpha_n |y - y^*| \end{aligned}\right\}, \tag{2.2}$$

where, $(x, y),\ (x, y^*) \in K$, and $0 \le \alpha_n < 1$ for all $n = 1, 2, \dots, N$. Define $\omega_n(x, y) = (L_n(x), F_n(x, y))$ for all $n = 1, 2, \dots, N$. The definition of a FIF originates from the following proposition.

Proposition 2.2 (see [3]). *The IFS $\{K; \omega_n, n = 1, 2, \dots, N\}$ has a unique attractor G such that G is the graph of a continuous function $f : I \to \mathbb{R}$ (called FIF associated with IFS $\{K; \omega_n, n = 1, 2, \dots, N\}$) satisfying $f(x_n) = y_n$ for $n = 0, 1, 2, \dots, N$.*

The following observations based on Proposition 2.2 are needed in the sequel. Let $\mathcal{F} = \{f : I \to \mathbb{R} \mid f$ is continuous, $f(x_0) = y_0$ and $f(x_N) = y_N\}$ and ρ be the sup-norm on $\mathcal{F}$. Then, $(\mathcal{F}, \rho)$ is a complete metric space. The FIF f is the unique fixed point of the Read–Bajraktarević operator T on $(\mathcal{F}, \rho)$ so that

$$Tf(x) \equiv F_n(L_n^{-1}(x), f(L_n^{-1}(x))) = f(x),\ x \in I_n, \quad n = 1, 2, \dots, N. \tag{2.3}$$

For an affine FIF, L_n and F_n are given by

$$\left.\begin{aligned} L_n(x) = a_n x + b_n \\ F_n(x, y) = \alpha_n y + q_n(x) \end{aligned}\right\}, \quad n = 1, 2, \dots, N, \tag{2.4}$$

where $q_n(x)$ is an affine map and $|\alpha_n| < 1$.

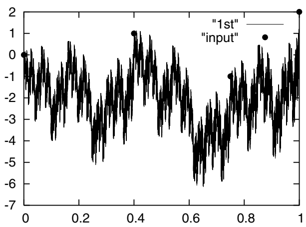
Barnsley and Harrington [1] have observed that the integral of a FIF is also a FIF, although for a different set of interpolation data, provided the value of the integral of the FIF at the initial endpoint of the interval is known. This observation is needed for developing the calculus of C^1-FIFs. Thus, let f be the FIF associated with $\{(L_n(x), F_n(x, y)),\ n = 1, 2, \dots, N\}$, where F_n is defined by (2.4) and let the value of integral of this FIF be known at x_0. If

$$\hat{f}(x) = \hat{y}_0 + \int_{x_0}^{x} f(\tau) d\tau, \tag{2.5}$$

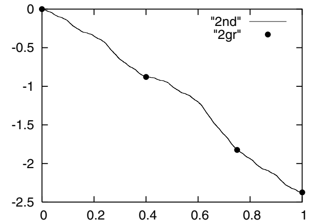
the function $\hat{f}$ is the FIF associated with IFS $\{(L_n(x), \hat{F}_n(x, y)),\ n = 1, 2, \dots, N\}$, where $\hat{F}_n(x, y) = a_n \alpha_n y + \hat{q}_n(x)$, $\hat{q}_n(x) = \hat{y}_{n-1} - a_n \alpha_n \hat{y}_0 + a_n \int_{x_0}^{x} q_n(\tau) d\tau$,

$$\hat{y}_n = \hat{y}_0 + \sum_{i=1}^{n} a_i \left\{ \alpha_i (\hat{y}_N - \hat{y}_0) + \int_{x_0}^{x_N} q_i(\tau) d\tau \right\}, \quad n = 1, 2, \dots, N-1,$$

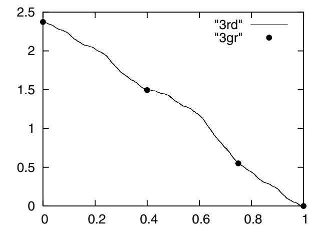
and $\hat{y}_N = \hat{y}_0 + \sum_{i=1}^{N} a_i \int_{x_0}^{x_N} q_i(\tau) d\tau / 1 - \sum_{i=1} N a_i \alpha_i$. Here, $(x_n, \hat{y}_n), n = 0, 1, 2, \dots, N$ are interpolation points of FIF $\hat{f}$.



(a) FIF f.



(b) $\hat{f}$ with $\hat{y}_0 = 0$.



(c) $\hat{f}$ with $\hat{y}_N = 0$.

FIG. 1. *FIF and its integrals.*

Remarks. 1. If the value of the integral of a FIF is known at the final endpoint x_N instead of the initial endpoint x_0, an analogue of the above result can be found by defining

$$\hat{f}(x) = \hat{y}_N - \int_x^{x_N} f(\tau)d\tau. \tag{2.6}$$

The function $\hat{f}$ is the FIF associated with $\{(L_n(x), \hat{F}_n(x,y)),\ n = 1, 2, \ldots, N\}$, where $\hat{F}_n(x,y) = a_n\alpha_n y + \hat{q}_n(x)$, $\hat{q}_n(x) = \hat{y}_n - a_n\alpha_n\hat{y}_N - a_n\int_x^{x_N} q_n(\tau)d\tau$ and the interpolation points of $\hat{f}$ are given by $\hat{y}_n = \hat{y}_N - \sum_{i=n+1}^N a_i\{\alpha_i(\hat{y}_N - \hat{y}_0) + \int_{x_0}^{x_N} q_i(\tau)d\tau\}$, $n = 1, 2, \ldots, N-1$ with $\hat{y}_0 = \hat{y}_N - \frac{\sum_{i=1}^N a_i \int_{x_0}^{x_N} q_i(\tau)d\tau}{1-\sum_{i=1}^N a_i\alpha_i}$. In general, a C^r-FIF interpolating a certain different set of data can be constructed when values of r successive integrals of the FIF are provided at any combination of endpoints.

2. The functional values of FIF $\hat{f}$ are, in general, different for the same set of vertical scaling factors even if $\hat{y}_0$ and $\hat{y}_N$ occurring, respectively, in (2.5) and (2.6) are the same. However, since $\hat{y}_n - \hat{y}_{n-1}$ remains the same for each n in both the cases, the nature of $\hat{f}$ remains the same in both the cases as illustrated by the following example.

Example. Let f be a FIF associated with the data $\{(0,0),\ (\frac{2}{5}, 1),\ (\frac{3}{4}, -1),\ (1, 2)\}$ with vertical scaling factor $\alpha_n = 0.8$ for $n = 1, 2, 3$ (Figure 1(a)). Choosing $\hat{y}_0 = 0$, $\hat{f}(x) = \int_{x_0}^x f$ interpolates the set of points $\{(0,0), (\frac{2}{5}, \frac{-22}{25}), (\frac{3}{4}, \frac{-73}{40}), (1, \frac{-19}{8})\}$. FIF $\hat{f}$ is associated with the IFS generated by $L_1(x) = \frac{2}{5}x,\ L_2(x) = \frac{7}{20}x + \frac{2}{5},\ L_3(x) = \frac{1}{4}x + \frac{3}{4}$ and $\hat{F}_1(x,y) = \frac{8}{25}y - \frac{3}{25}x^2,\ \hat{F}_2(x,y) = \frac{7}{25}y - \frac{63}{100}x^2 + \frac{7}{20}x - \frac{22}{25},\ \hat{F}_3(x,y) = \frac{1}{5}y + \frac{7}{40}x^2 - \frac{1}{4}x + \frac{73}{40}$. The graph of FIF $\hat{f}$ is shown in Figure 1(b). Next, choosing $\hat{y}_N = 0$, $\hat{f}(x) = -\int_x^{x_N} f$ interpolates the set of points $\{(0, \frac{19}{8}), (\frac{2}{5}, \frac{299}{200}), (\frac{3}{4}, \frac{11}{20}), (1, 0)\}$ (Figure 1(c)). In this case, the corresponding IFS contains the same $L_n(x)$ for $n = 1, 2, 3$ and $\hat{F}_1(x,y) = \frac{8}{25}y - \frac{3}{25}x^2 + \frac{323}{100},\ \hat{F}_2(x,y) = \frac{7}{25}y - \frac{63}{100}x^2 + \frac{7}{20}x + \frac{83}{100},\ \hat{F}_3(x,y) = \frac{1}{5}y + \frac{7}{40}x^2 - \frac{1}{4}x + \frac{3}{40}$. The nature of FIFs $\hat{f}$ in Figure 1(b)–(c) remains the same, since the functional values of FIF $\hat{f}$ in Figure 1(c) are shifted by $\frac{19}{8}$ from the functional values of $\hat{f}$ in Figure 1(b) so that $\hat{y}_n - \hat{y}_{n-1}$ remains the same. It is interesting to note that the corresponding functions $\hat{F}_n(x,y)$ for IFS of Figure 1(b)–(c) are not shifted by equal amount although the function $\hat{f}$ is shifted by the fixed amount $\frac{19}{8}$.

In general, the relation between the IFS of FIF f and the IFS of its integral $\hat{f}$ is given as follows [18].

PROPOSITION 2.3. *Let $\hat{f}$ be the FIF defined by* (2.5) *or* (2.6) *for a FIF f with $L_n(x)$ and $F_n(x,y)$ given by* (2.4). *Then, f is primitive of $\hat{f}$ if and only if $\hat{f}$ is the FIF associated with the IFS $\{\mathbb{R}^2; \hat{w}_n(x,y) = (L_n(x), \hat{F}_n(x,y)), n = 1, 2, \ldots, N\}$, where*

$\hat{F}_n(x,y) = \hat{\alpha}_n y + \hat{q}_n(x)$, $\hat{\alpha}_n = a_n\alpha_n$, *and the polynomial* $\hat{q}_n(x)$ *satisfies* $\hat{q}'_n = a_n q_n$ *for* $n = 1, 2, \ldots, N$.

2.2. Principle of construction of a C^r-FIF. Our approach for the construction of a C^r-FIF that interpolates the given data is based on finding the solution of a system of equations in which any type of boundary conditions are admissible. Such a construction is more general than that of Barnsley and Harrington [1] wherein all the relevant derivatives of the FIF are restricted to be known at the initial endpoint only. The C^r-FIF interpolating prescribed set of data is found as the fixed point of a suitably chosen IFS by using the following procedure.

Let $\{(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)\}$, $x_0 < x_1 < \cdots < x_N$, be the given data points and $\mathcal{F}^r = \{g \in C^r(I, \mathbb{R}) \mid g(x_0) = y_0 \text{ and } g(x_N) = y_N\}$, where r is some nonnegative integer and σ is the C^r-norm on $\mathcal{F}^r$. Define the Read–Bajraktarević operator T on $(\mathcal{F}^r, \sigma)$ as

$$Tg(x) = \alpha_n g(L_n^{-1}(x)) + q_n(L_n^{-1}(x)),\ x \in I_n, \quad n = 1, 2, \ldots, N,$$

where $L_n(x) = a_n x + b_n$ satisfies (2.1), $q_n(x)$ is a suitably chosen polynomial, and $|\alpha_n| < a_n^r$ for $n = 1, 2, \ldots, N$. The condition $|\alpha_n| < a_n^r < 1$ gives that T is a contractive operator on $(\mathcal{F}^r, \sigma)$. The fixed point f of T is a FIF that satisfies the functional relation, $f(L_n(x)) = \alpha_n f(x) + q_n(x)$ for $n = 1, 2, \ldots, N$. Using Proposition 2.3, it follows that f' satisfies the functional relation

$$f'(L_n(x)) = \frac{\alpha_n f'(x) + q'_n(x)}{a_n}, \quad n = 1, 2, \ldots, N.$$

Since $\frac{|\alpha_n|}{a_n} \le \frac{|\alpha_n|}{a_n^r} < 1$, f' is a fractal function. Inductively, using the above arguments, the following relations are obtained:

$$f^{(k)}(L_n(x)) = \frac{\alpha_n f^{(k)}(x) + q_n^{(k)}(x)}{a_n^k}, \quad n = 1, 2, \ldots, N, \quad k = 0, 1, 2, \ldots, r, \tag{2.7}$$

where $f^{(0)} = f$ and $q^{(0)} = q$. Since $\frac{|\alpha_n|}{a_n^k} \le \frac{|\alpha_n|}{a_n^r} < 1$, the derivatives $f^{(k)}$, $k = 2, 3, \ldots, r$ are fractal functions. In general $f^{(k)}$, $k = 1, 2, 3, \ldots, r$, interpolates a data different than the given data. In particular, $f^{(r)}$ is an affine FIF if the polynomial $q_n^{(r)}$ occurring in (2.7) with $k = r$ is affine. Thus, $q_n(x)$ is chosen as a polynomial of degree $(r+1)$. Let $q_n(x) = \sum_{k=0}^{r+1} q_{kn} x^k$, $n = 1, 2, \ldots, N$, where the coefficients q_{kn} are chosen suitably such that f interpolates the prescribed data. The continuity of $f^{(k)}$ on I implies

$$f^{(k)}(L_{n+1}(x_0)) = f^{(k)}(L_n(x_N)),\ \ k = 0, 1, \ldots, r, \quad n = 1, 2, \ldots, N-1.$$

Therefore, (2.7) results in the following $(r+1)(N-1)$ join-up conditions for $k = 0, 1, \ldots, r$, $n = 1, 2, \ldots, N-1$:

$$\frac{\alpha_{n+1} f^{(k)}(x_0) + q_{n+1}^{(k)}(x_0)}{a_{n+1}^k} = \frac{\alpha_n f^{(k)}(x_N) + q_n^{(k)}(x_N)}{a_n^k}. \tag{2.8}$$

In addition, at the endpoints of the interval, (2.7) implies that the values of $f^{(k)}$ satisfy the following $2r$-conditions:

$$f^{(k)}(x_0) = \frac{\alpha_1 f^{(k)}(x_0) + q_1^{(k)}(x_0)}{a_1^k}, \quad k = 1, 2, \ldots, r, \tag{2.9}$$

and

$$f^{(k)}(x_N) = \frac{\alpha_N f^{(k)}(x_N) + q_N^{(k)}(x_N)}{a_N^k}, \quad k = 1, 2, \ldots, r. \tag{2.10}$$

Let the prescribed interpolation conditions be

$$f(x_n) = y_n, \quad n = 0, 1, \ldots, N. \tag{2.11}$$

In view of (2.8)–(2.11), the total number of conditions for f to interpolate the given data are $(r+1)(N-1) + 2r + (N+1) = (r+2)N + r$. In (2.8)–(2.10), $f^{(k)}(x_0)$ and $f^{(k)}(x_N)$ for $k = 1, 2, \ldots, r$ are $2r$ unknowns and $q_{kn},\ k = 0, 1, \ldots, r+1,\ n = 1, 2, \ldots, N$, in the polynomials $q_n(x)$ are additional $(r+2)N$ unknowns. Consequently, in total $(r+2)N+2r$ number of unknowns are to be determined. The principle of construction of a C^r-FIF is to determine these unknowns by choosing additional suitable r conditions in the form of restrictions on the values of the C^r-FIF or the values of its derivatives at the boundary points of $[x_0, x_N]$ such that (2.8)–(2.11) together with these additional conditions are linearly independent. The above unknowns are determined uniquely as the solution of these linear independent system of equations. Thus, the desired C^r-FIF f interpolating the given data is constructed as the attractor of the following IFS:

$$\{\mathbb{R}^2; \omega_n(x, y) = (L_n(x), F_n(x, y) = \alpha_n y + q_n(x)), n = 1, 2, \ldots, N\},$$

where $|\alpha_n| < a_n^r$ and $q_n(x)$, $n = 1, 2, \ldots, N$, are the polynomials with coefficients q_{kn} computed by solving the linear independent system of equations, given by the above procedure. The flexibility of these choices of boundary conditions allows for the construction of a wide range of spline FIFs. Even for a given choice of boundary conditions, depending upon the nature of the problem or simply at the discretion of the user, an infinite number of suitable spline FIFs may be constructed due to the freedom of choices for vertical scaling factors in our construction.

Remarks. 1. Barnsley and Harrington's construction [1] of a C^r-FIF f is done by restricting the choice of boundary values $f^{(k)}(x)$ for $k = 1, 2, \ldots, r$, at the initial endpoint. In our above construction of C^r-FIFs, all kinds of boundary conditions are admissible.

2. It seems that Barnsley and Harrington's question—"whether there exists a FIF as a fixed point of an IFS wherein horizontal scalings are allowed in the reverse direction"—is raised [1], since the construction of a C^r-FIF is based upon restricting boundary values of $f^{(k)}$ at only initial end point of I. Such a question does not arise in our construction since the boundary values of $f^{(k)}$ for C^r-FIF f are admissible at any combination of boundary points of I.

Since the classical cubic splines play a significant role in CAGD, surface analysis, differential equation, FEM, and other applications (see, e.g., [13, 14, 15]), in the sequel a detailed construction for such cubic spline FIFs based on the above approach is given in the following section.

3. Construction of cubic spline FIFs through moments. In the present section, cubic spline FIFs f_Δ are constructed through the *moments* $M_n = f''_\Delta(x_n)$ for $n = 0, 1, 2, \ldots, N$.

DEFINITION 3.1. *A function* $f_\Delta(x) \equiv f_\Delta(Y; x)$ *is called a cubic spline FIF interpolating a set of ordinates* $Y : y_0, y_1, y_2, \ldots, y_N$ *with respect to the mesh* Δ :

$x_0 < x_1 < x_2 < \cdots < x_N$ *if* (i) $f_\Delta \in C^2[x_0, x_N]$, (ii) f_Δ *satisfies the interpolation conditions* $f_\Delta(x_n) = y_n$, $n = 0, 1, \dots, N$ *and* (iii) *the graph of* f_Δ *is fixed point of a IFS,* $\{\mathbb{R}^2; \omega_n(x, y), n = 1, 2, \dots, N\}$, *where for* $n = 1, 2, \dots, N$, $\omega_n(x, y) = (L_n(x), F_n(x, y))$, $L_n(x)$ *is defined by* (2.4), $F_n(x, y) = a_n^2 \alpha_n y + a_n^2 q_n(x)$, $0 < |\alpha_n| < 1$, *and* $q_n(x)$ *is a suitable cubic polynomial.*

Using the moments M_n, $n = 0, 1, 2, \dots, N$, a rectangular system of equations is formed for determining the polynomial $q_n(x)$ by employing the following procedure.

Using property (iii) and (2.3), it follows that f''_Δ satisfies the functional equation

$$f''_\Delta(L_n(x)) = \alpha_n f''_\Delta(x) + \frac{c_n(x - x_0)}{x_N - x_0} + d_n, \quad n = 1, 2, \dots, N. \tag{3.1}$$

By (2.1) and (3.1), $c_n = M_n - M_{n-1} - \alpha_n(M_N - M_0)$ and $d_n = M_{n-1} - \alpha_n M_0$. Thus, for $n = 1, 2, \dots, N$, (3.1) can be rewritten as

$$f''_\Delta(L_n(x)) = \alpha_n f_\Delta''(x) + \frac{(M_n - \alpha_n M_N)(x - x_0)}{x_N - x_0} + \frac{(M_{n-1} - \alpha_n M_0)(x_N - x)}{x_N - x_0}. \tag{3.2}$$

The function f''_Δ being continuous on I could be twice integrated to obtain

$$\begin{aligned} f_\Delta(L_n(x)) = a_n^2 \Big\{ &\alpha_n f_\Delta(x) + \frac{(M_n - \alpha_n M_N)(x - x_0)^3}{6(x_N - x_0)} + \frac{(M_{n-1} - \alpha_n M_0)(x_N - x)^3}{6(x_N - x_0)} \\ &+ c_n^*(x_N - x) + d_n^*(x - x_0) \Big\}, \quad n = 1, 2, \dots, N. \end{aligned} \tag{3.3}$$

Now using interpolation conditions and (2.1), the constants c_n^* and d_n^* are determined as

$$\begin{aligned} c_n^* &= \frac{1}{x_N - x_0}\left(\frac{y_{n-1}}{a_n^2} - \alpha_n y_0\right) - \frac{(M_{n-1} - \alpha_n M_0)(x_N - x_0)}{6}, \\ d_n^* &= \frac{1}{x_N - x_0}\left(\frac{y_n}{a_n^2} - \alpha_n y_N\right) - \frac{(M_n - \alpha_n M_N)(x_N - x_0)}{6}. \end{aligned}$$

Thus, the functional equation (3.3) for the cubic spline FIF in terms of moments can be written as

$$\begin{aligned} f_\Delta(L_n(x)) = a_n^2 \Big\{ &\alpha_n f_\Delta(x) + \frac{(M_n - \alpha_n M_N)(x - x_0)^3}{6(x_N - x_0)} + \frac{(M_{n-1} - \alpha_n M_0)(x_N - x)^3}{6(x_N - x_0)} \\ &- \frac{(M_{n-1} - \alpha_n M_0)(x_N - x_0)(x_N - x)}{6} - \frac{(M_n - \alpha_n M_N)(x_N - x_0)(x - x_0)}{6} \\ &+ \left(\frac{y_{n-1}}{a_n^2} - \alpha_n y_0\right)\frac{x_N - x}{x_N - x_0} + \left(\frac{y_n}{a_n^2} - \alpha_n y_N\right)\frac{x - x_0}{x_N - x_0} \Big\}, \quad n = 1, 2, \dots, N. \end{aligned} \tag{3.4}$$

It follows by (3.4) that $f_\Delta(x)$ is continuous on $[x_0, x_N]$ and satisfies the interpolating conditions $f_\Delta(x_n) = y_n$, $n = 0, 1, 2, \dots, N$. Further, (3.4) gives that, on $[x_{i-1}, x_i]$,

$i = 1, 2, \ldots, N,$

(3.5)
$$f'_\Delta(L_i(x)) = a_i\Big\{\alpha_i f'_\Delta(x) + \frac{(M_i - \alpha_i M_N)(x-x_0)^2}{2(x_N - x_0)} - \frac{(M_{i-1} - \alpha_i M_0)(x_N - x)^2}{2(x_N - x_0)} - \frac{[M_i - M_{i-1} - \alpha_i(M_N - M_0)](x_N - x_0)}{6} + \Big[\frac{y_i - y_{i-1}}{a_i^2} - \alpha_i(y_N - y_0)\Big]\frac{1}{x_N - x_0}\Big\}.$$

Denote $x_n - x_{n-1}$ by $h_n =$ for $n = 1, 2, \ldots, N$. Since, by property (i), $f'_\Delta(x)$ is continuous at $x_1, x_2, \ldots, x_{N-1}$, $\lim_{x\to x_n^-} f'_\Delta(x) = \lim_{x\to x_n^+} f'_\Delta(x)$, $n = 1, 2, \ldots, N-1$. Thus, using (3.5) for $i = n$ and $i = n+1$, we have

(3.6)
$$\begin{aligned} -a_{n+1}\alpha_{n+1} f'_\Delta(x_0) &- \frac{\alpha_n h_n + 2\alpha_{n+1}h_{n+1}}{6} M_0 + \frac{h_n}{6} M_{n-1} + \frac{h_n + h_{n+1}}{3} M_n \\ &+ \frac{h_{n+1}}{6} M_{n+1} - \frac{2\alpha_n h_n + \alpha_{n+1}h_{n+1}}{6} M_N + a_n\alpha_n f'_\Delta(x_N) \\ &= \frac{y_{n+1} - y_n}{h_{n+1}} - \frac{y_n - y_{n-1}}{h_n} - (a_{n+1}\alpha_{n+1} - a_n\alpha_n)\frac{y_N - y_0}{x_N - x_0}, \quad n = 1, 2, \ldots, N-1. \end{aligned}$$

Introducing the notations,

$$A_n^* = \frac{-6a_{n+1}\alpha_{n+1}}{h_n + h_{n+1}},\ A_n = \frac{-(\alpha_n h_n + 2\alpha_{n+1}h_{n+1})}{h_n + h_{n+1}},\ \lambda_n = \frac{h_{n+1}}{h_n + h_{n+1}},$$
$$\mu_n = 1 - \lambda_n,\ B_n = \frac{-(2\alpha_n h_n + \alpha_{n+1}h_{n+1})}{h_n + h_{n+1}},\ B_n^* = \frac{6a_n\alpha_n}{h_n + h_{n+1}},$$

for $n = 1, 2, \ldots, N-1$, the continuity relation (3.6) reduces to

$$(3.7)\qquad \begin{aligned} &A_n^* f'_\Delta(x_0) + A_n M_0 + \mu_n M_{n-1} + 2M_n + \lambda_n M_{n+1} + B_n M_N + B_n^* f'_\Delta(x_N) \\ &\quad = \frac{6[(y_{n+1} - y_n)/h_{n+1} - (y_n - y_{n-1})/h_n]}{h_n + h_{n+1}} - \frac{6(a_{n+1}\alpha_{n+1} - a_n\alpha_n)}{h_n + h_{n+1}}\frac{y_N - y_0}{x_N - x_0}. \end{aligned}$$

Next, (3.5) with $x = x_0$ and $i = 1$ gives the following functional relation for $f'_\Delta(x_0)$:

$$(3.8)\qquad \begin{aligned} &6(1 - a_1\alpha_1) f'_\Delta(x_0) + 2(1 - \alpha_1)h_1 M_0 + h_1 M_1 - \alpha_1 h_1 M_N \\ &\quad = 6/h_1[y_1 - y_0 - \alpha_1 a_1^2(y_N - y_0)]. \end{aligned}$$

Similarly, (3.5) with $x = x_N$ and $i = N$ gives

$$(3.9)\qquad \begin{aligned} &-\alpha_N h_N M_0 + h_N M_{N-1} + 2(1 - \alpha_N)h_N M_N - 6(1 - a_N\alpha_N) f'_\Delta(x_N) \\ &\quad = -6/h_N[y_N - y_{N-1} - \alpha_N a_N^2(y_N - y_0)]. \end{aligned}$$

To write the system of equations given by (3.7)–(3.9) in matrix from, we introduce the following notations:

$$\begin{aligned} A_0^* &= 6(1 - a_1\alpha_1),\ A_0 = 2(1 - \alpha_1)h_1,\ \lambda_0 = h_1,\ B_0 = -\alpha_1 h_1, \\ A_N &= -\alpha_N h_N,\ \mu_N = h_N,\ B_N = 2(1 - \alpha_N)h_N,\ B_N^* = -6(1 - a_N\alpha_N), \\ d_0 &= 6/h_1[y_1 - y_0 - \alpha_1 a_1^2(y_N - y_0)],\ d_N = -6/h_N[y_N - y_{N-1} - \alpha_N a_N^2(y_N - y_0)]. \end{aligned}$$

Thus, the matrix form of defining (3.7)–(3.9) is

(3.10)

$$\begin{bmatrix} A_0^* & A_0 & \lambda_0 & 0 & 0 & \dots & 0 & 0 & 0 & B_0 & 0 \\ A_1^* & A_1+\mu_1 & 2 & \lambda_1 & 0 & \dots & 0 & 0 & 0 & B_1 & B_1^* \\ A_2^* & A_2 & \mu_2 & 2 & \lambda_2 & \dots & 0 & 0 & 0 & B_2 & B_2^* \\ A_3^* & A_3 & 0 & \mu_3 & 2 & \dots & 0 & 0 & 0 & B_3 & B_3^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{N-3}^* & A_{N-3} & 0 & 0 & 0 & \dots & 2 & \lambda_{N-3} & 0 & B_{N-3} & B_{N-3}^* \\ A_{N-2}^* & A_{N-2} & 0 & 0 & 0 & \dots & \mu_{N-2} & 2 & \lambda_{N-2} & B_{N-2} & B_{N-2}^* \\ A_{N-1}^* & A_{N-1} & 0 & 0 & 0 & \dots & 0 & \mu_{N-1} & 2 & \lambda_{N-1}+B_{N-1} & B_{N-1}^* \\ 0 & A_N & 0 & 0 & 0 & \dots & 0 & 0 & \mu_N & B_N & B_N^* \end{bmatrix} \begin{bmatrix} f'_\Delta(x_0) \\ M_0 \\ M_1 \\ M_2 \\ \\ \vdots \\ M_{N-2} \\ M_{N-1} \\ M_N \\ f'_\Delta(x_N) \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ \\ \vdots \\ d_{N-3} \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix},$$

where $d_n,\ n = 1, 2, \dots, N-1$, is given by the right side expression of (3.7) and $f'_\Delta(x_0), M_0,\ M_1, \dots, M_N, f'_\Delta(x_N)$ are unknowns. Equation (3.10), consisting of a coefficient matrix of order $(N+1)\times(N+3)$, is the desired rectangular matrix equation for computing the unknowns coefficients q_{kn} of the polynomial $q_n(x)$.

Boundary Conditions. By prescribing suitable boundary conditions as in the case of classical cubic splines, the rectangular matrix system of equations (3.10) reduces to a square matrix system of equations. Let the data $\{(x_n, y_n) : n = 0, 1, 2, \dots, N\}$ be generated by a continuous function Φ that is to be approximated by cubic spline FIF f_Δ. The following kinds of boundary conditions are admissible.

Boundary conditions of Type-I: In this case, the values of the first derivative are prescribed at the endpoints of the interval $[x_0, x_N]$, i.e., $f'_\Delta(x_0) = \Phi'(x_0), \quad f'_\Delta(x_N) = \Phi'(x_N)$. So, (3.10) reduces to the following system of equations:

$$\begin{bmatrix} A_0 & \lambda_0 & 0 & 0 & \dots & 0 & 0 & 0 & B_0 \\ A_1+\mu_1 & 2 & \lambda_1 & 0 & \dots & 0 & 0 & 0 & B_1 \\ A_2 & \mu_2 & 2 & \lambda_2 & \dots & 0 & 0 & 0 & B_2 \\ A_3 & 0 & \mu_3 & 2 & \dots & 0 & 0 & 0 & B_3 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ A_{N-3} & 0 & 0 & 0 & \dots & 2 & \lambda_{N-3} & 0 & B_{N-3} \\ A_{N-2} & 0 & 0 & 0 & \dots & \mu_{N-2} & 2 & \lambda_{N-2} & B_{N-2} \\ A_{N-1} & 0 & 0 & 0 & \dots & 0 & \mu_{N-1} & 2 & \lambda_{N-1}+B_{N-1} \\ A_N & 0 & 0 & 0 & \dots & 0 & 0 & \mu_N & B_N \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ \\ \vdots \\ M_{N-3} \\ M_{N-2} \\ M_{N-1} \\ M_N \end{bmatrix} = \begin{bmatrix} d_0^1 \\ d_1^1 \\ d_2^1 \\ \\ \vdots \\ d_{N-2}^1 \\ d_{N-1}^1 \\ d_N^1 \end{bmatrix}, \tag{3.11}$$

where $d_0^1 = d_0 - A_0^* f'_\Delta(x_0),\ d_n^1 = d_n - A_n^* f'_\Delta(x_0) - B_n^* f'_\Delta(x_N)$ for $n = 1, 2, \dots, N-1$, and $d_N^1 = d_N - B_N^* f'_\Delta(x_N)$. Thus, boundary conditions of Type-I result in determination of the *complete cubic spline FIF* by using (3.11).

Boundary conditions of Type-II: In this case, the values of the second derivative given at the endpoints of the segment $[x_0, x_N]$ are prescribed as $f''_\Delta(x_0) = \Phi''(x_0) = M_0, f''_\Delta(x_N) = \Phi''(x_N) = M_N$. With these boundary conditions, (3.10) reduces to

$$\begin{bmatrix} A_0^* & \lambda_0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ A_1^* & 2 & \lambda_1 & 0 & \dots & 0 & 0 & 0 & B_1^* \\ A_2^* & \mu_2 & 2 & \lambda_2 & \dots & 0 & 0 & 0 & B_2^* \\ A_3^* & 0 & \mu_3 & 2 & \dots & 0 & 0 & 0 & B_3^* \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ A_{N-3}^* & 0 & 0 & 0 & \dots & 2 & \lambda_{N-3} & 0 & B_{N-3}^* \\ A_{N-2}^* & 0 & 0 & 0 & \dots & \mu_{N-2} & 2 & \lambda_{N-2} & B_{N-2}^* \\ A_{N-1}^* & 0 & 0 & 0 & \dots & 0 & \mu_{N-1} & 2 & B_{N-1}^* \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \mu_N & B_N^* \end{bmatrix} \begin{bmatrix} f'_\Delta(x_0) \\ M_1 \\ M_2 \\ M_3 \\ \\ \vdots \\ M_{N-3} \\ M_{N-2} \\ M_{N-1} \\ f'_\Delta(x_N) \end{bmatrix} = \begin{bmatrix} d_0^2 \\ d_1^2 \\ d_2^2 \\ \\ \vdots \\ d_{N-2}^2 \\ d_{N-1}^2 \\ d_N^2 \end{bmatrix}, \tag{3.12}$$

where $d_1^2 = d_1 - (A_1+\mu_1)M_0 - B_1\mu_N,\ d_{N-1}^2 = d_{N-1} - A_{N-1}M_0 - (B_{N-1}+\lambda_{N-1})M_N$, and $d_n^2 = d_n - A_nM_0 - B_nM_N$ for $n = 0, 2, 3, \dots, N-2, N$. Taking free end conditions $M_0 = 0$ and $M_N = 0$, the *natural cubic spline FIF* is computed by using (3.12).

Boundary conditions of Type-III: In this case, the boundary conditions involve the functional values, the values of first and second derivatives of the cubic splines at both endpoints, i.e., $f_\Delta(x_0) = f_\Delta(x_N)$, $f'_\Delta(x_0) = f'_\Delta(x_N)$, $f''_\Delta(x_0) = f''_\Delta(x_N)$. With these boundary conditions, (3.10) takes the following form:

(3.13)

$$\begin{bmatrix} A_0^* & \lambda_0 & 0 & 0 & \dots & 0 & 0 & 0 & A_0+B_0 \\ A_1^*+B_1^* & 2 & \lambda_1 & 0 & \dots & 0 & 0 & 0 & A_1+B_1+\mu_1 \\ A_2^*+B_2^* & \mu_2 & 2 & \lambda_2 & \dots & 0 & 0 & 0 & A_2+B_2 \\ A_3^*+B_3^* & 0 & \mu_3 & 2 & \dots & 0 & 0 & 0 & A_3+B_3 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ A_{N-3}^*+B_{N-3}^* & 0 & 0 & 0 & \dots & 2 & \lambda_{N-3} & 0 & A_{N-3}+B_{N-3} \\ A_{N-2}^*+B_{N-2}^* & 0 & 0 & 0 & \dots & \mu_{N-2} & 2 & \lambda_{N-2} & A_{N-2}+B_{N-2} \\ A_{N-1}^*+B_{N-1}^* & 0 & 0 & 0 & \dots & 0 & \mu_{N-1} & 2 & A_{N-1}+B_{N-1}+\lambda_{N-1} \\ B_N^* & 0 & 0 & 0 & \dots & 0 & 0 & \mu_N & A_N+B_N \end{bmatrix} \begin{bmatrix} f'_\Delta(x_0) \\ M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{N-3} \\ M_{N-2} \\ M_{N-1} \\ M_N \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{N-3} \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix}.$$

The *periodic cubic spline FIF* is computed by using (3.13).

We confine ourselves to the boundary conditions of Type-I, Type-II, and Type-III only for the convergence results in section 4 although, in addition to the above kinds of boundary conditions, the following types of boundary conditions are also admissible in our approach.

Boundary conditions of Type-IV: In this case, the values of derivatives of given function are known at either initial or final endpoint of the interval, i.e., $f'_\Delta(x_0) = \Phi'(x_0)$, $f''_\Delta(x_0) = \Phi''(x_0) = M_0$ or $f'_\Delta(x_N) = \Phi'(x_N)$, $f''_\Delta(x_N) = \Phi''(x_N) = M_N$. Barnsley and Harrington [1] used the former set of conditions to obtain the cubic spline FIF by employing an involved algebraic method.

Boundary conditions of Type-V: In this type of boundary condition, two sets of conditions are possible depending on the values of different order of the derivatives at both endpoints, i.e., $f'_\Delta(x_0) = \Phi'(x_0)$, $f''_\Delta(x_N) = \Phi''(x_N) = M_N$ or $f'_\Delta(x_N) = \Phi'(x_N)$, $f_\Delta''(x_0) = \Phi''(x_0) = M_0$. In order to find the respective unknowns, the square matrix of order $(N+1)$ for the boundary conditions of Type-IV and Type-V can be obtained from (3.10).

Boundary conditions of Type-VI: Two linear equations involving M_0, $f'_\Delta(x_0)$, $f'_\Delta(x_N)$, and M_N are considered in this case such that these and (3.10) form a linearly independent system of equations. The resulting square matrix of order $(N+3)$ can be solved to find all $(N+3)$ unknowns simultaneously.

Using one of the above types of boundary conditions and solving the corresponding system of equations, the values $f'_\Delta(x_0), M_0, M_1, \dots, M_N$ and $f'_\Delta(x_N)$ are determined. These values of M_n, $n = 0, 1, 2, \dots, N$, are used in the construction of an associated IFS given by

$$\{\mathbb{R}^2; \omega_n(x,y) = (L_n(x), F_n(x,y)), n = 1, 2, \dots, N\}, \tag{3.14}$$

where $L_n(x) = a_n x + b_n$ and

(3.15)

$$\begin{aligned} F_n(x,y) = a_n^2 \Bigg\{ & \alpha_n f_\Delta(x) + \frac{(M_n - \alpha_n M_N)(x - x_0)^3}{6(x_N - x_0)} + \frac{(M_{n-1} - \alpha_n M_0)(x_N - x)^3}{6(x_N - x_0)} \\ & - \frac{(M_{n-1} - \alpha_n M_0)(x_N - x_0)(x_N - x)}{6} - \frac{(M_n - \alpha_n M_N)(x_N - x_0)(x - x_0)}{6} \\ & + \left(\frac{y_{n-1}}{a_n^2} - \alpha_n y_0\right) \frac{x_N - x}{x_N - x_0} + \left(\frac{y_n}{a_n^2} - \alpha_n y_N\right) \frac{x - x_0}{x_N - x_0} \Bigg\}. \end{aligned}$$

The graph of the desired cubic spline is the fixed point of the IFS given by (3.14).

Remarks. 1. If the vertical scaling factor $\alpha_n = 0$ for $n = 1, 2, \ldots, N$, $F_n(x, y)$ reduces to a cubic polynomial in each subinterval of I so that in this case the resulting FIF is a classical cubic spline.

2. By the fixed point theorem, with prescribed ordinates at mesh points, the nonperiodic spline FIF always exists and is unique for a given choice of vertical scaling factors. This spline FIF has simple end supports ($M_0 = 0, M_N = 0$), prescribed end moments or simple supports at points beyond mesh extremities. Similarly, the periodic spline FIF exists and is unique for a given data and a given choice of vertical scaling factors. Since the moments depend upon the vertical scaling factors α_n, by changing α_n, infinitely many nonperiodic splines or periodic splines having the same boundary conditions can be constructed. This gives an additional advantage for the applications of the cubic spline FIF over the applications of the classical cubic spline since there is no flexibility in choosing the latter once the boundary conditions are fixed.

3. Clearly, the replacement of y_n by $y_n + c$ does not affect the right-hand sides of (3.7)–(3.9). Thus, $f_\Delta(Y; x) + \eta = f_\Delta(\bar{Y}; x)$, where $\bar{Y} : \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_N$ and $\bar{y}_n = y_n + \eta$, $n = 0, 1, 2, \ldots, N$, with η being a constant. Since the moments M_n do not change by the translation of the ordinates by a constant η, it follows that it is possible to associate more than one cubic spline FIF for a given set of moments M_n. This property of cubic spline FIF f_Δ is analogous to the corresponding property of the periodic classical spline [19].

4. The existence of spline FIF f_Δ gives (3.7)–(3.9). Further, if spline FIF f_Δ is periodic, adding (3.7) to (3.9) gives

$$\sum_{n=1}^{N} [(h_n + h_{n+1})M_n - 2\alpha_n h_n M_N] = 0. \tag{3.16}$$

The condition (3.16) is therefore a necessary condition for the existence of the periodic cubic spline FIF for prescribed moments M_n. With $\alpha_n = 0$ for $n = 1, 2, \ldots, N$, the condition (3.16) reduces to the necessary condition for the existence of periodic classical cubic spline associated with M_n [19, p. 17].

5. For a prescribed set of data and a suitable choice of α_n satisfying $0 \leq |\alpha_n| < 1$, it follows from (3.15) that, on the space $\mathcal{F}^* = \{f \in C^2(I, \mathbb{R}) \mid f(x_0) = y_0 \text{ and } f(x_N) = y_N\}$, cubic spline FIF f_Δ is the fixed point of Read–Bajraktarević operator T^* defined by

$$\begin{aligned} T^* f(x) = a_n^2 \Bigg\{ &\alpha_n f(L_n^{-1}(x)) + \frac{(M_n - \alpha_n M_N)(L_n^{-1}(x) - x_0)^3}{6(x_N - x_0)} \\ &+ \frac{(M_{n-1} - \alpha_n M_0)(x_N - L_n^{-1}(x))^3}{6(x_N - x_0)} - \frac{(M_{n-1} - \alpha_n M_0)(x_N - x_0)(x_N - L_n^{-1}(x))}{6} \\ &- \frac{(M_n - \alpha_n M_N)(x_N - x_0)(L_n^{-1}(x) - x_0)}{6} \\ &+ \left(\frac{y_{n-1}}{a_n^2} - \alpha_n y_0\right) \frac{x_N - L_n^{-1}(x)}{x_N - x_0} + \left(\frac{y_n}{a_n^2} - \alpha_n y_N\right) \frac{L_n^{-1}(x) - x_0}{x_N - x_0} \Bigg\}, \end{aligned} \tag{3.17}$$

where $x \in I_n$ for $n = 1, 2, \ldots, N$. Since (3.10) is derived from the fixed point relation $T^* f_\Delta = f_\Delta$, the solution of each of the equations (3.11)–(3.13) is unique due to uniqueness of the fixed point. Hence, the coefficient matrices in the systems (3.11)–(3.13) are invertible.

6. The moment integral $\Phi_m = \int_I x^m \Phi(x)\ dx$, $m = 0, 1, 2, \ldots$, of the data generating function Φ can be approximately calculated by integral moments $f_\Delta^m \equiv \int_I x^m f_\Delta(x)\, dx$ of the cubic spline FIF. One can evaluate explicitly the moment integral f_Δ^m in terms of $f_\Delta^{m-1}, f_\Delta^{m-2}, \ldots, f_\Delta^0$, the data points, the vertical scaling factors $\alpha_n, n = 1, 2, \ldots, N$, and $Q_m = \int_I x^m Q(x)\ dx$, where $Q(x) = q_n \circ L_n^{-1}(x),\ x \in I_n$. Thus, Barnsley and Harrington's query [1] regarding the moment integrals in case of reverse horizontal scaling is already taken into account in our construction.

4. Convergence of cubic spline FIFs. Define a sequence $\{\Delta_k\}$ of meshes on $[x_0, x_N]$ as $\Delta_k : x_0 = x_{k,0} < x_{k,1} < \cdots < x_{k,N_k} = x_N$, then set $h_{k,n} = x_{k,n} - x_{k,n-1}$ and $\|\Delta_k\| = \max_{1 \le n \le N_k} h_{k,n}$.

We establish that sequences of cubic spline FIFs $\{f_{\Delta_k}(x)\}$ converge to $\Phi(x)$ on suitable sequences of meshes $\{\Delta_k\}$ at the rate of square of the mesh norm $\|\Delta_k\|$, where $\Phi \in C^r(I)$, $r = 2, 3$, or 4, is the data generating function. Since the matrices associated with the cubic spline FIF, satisfying the boundary conditions of Type-I, Type-II, or Type-III (periodic), are not, in general, diagonally dominant and $f_\Delta^{''}(x)$ is not piecewise linear, the convergence procedure for classical cubic spline [19] cannot be adopted for establishing the convergence of the cubic spline FIF. Our convergence results for cubic spline FIFs are in fact derived by using the convergence results for classical splines.

Let $\mathcal{F}^*$ be the set of cubic spline FIFs on the given mesh Δ, interpolating the values y_n at the mesh points. From (3.17), it is clear that for $x \in I = [x_0, x_N]$,

$$f_\Delta(L_n(x)) = a_n^2 \alpha_n f_\Delta(x) + a_n^2 q_n(x), \tag{4.1}$$

where $q_n(x)$ is a cubic polynomial for $n = 1, 2, \ldots,$N. Throughout the sequel, we assume $|\alpha_n| \le s < 1$ for a fixed s and denote $q_n(\alpha_n, x) \equiv q_n(x)$ for $n = 1, 2, \ldots, N$.

LEMMA 4.1. *Let $f_\Delta(x)$ and $S_\Delta(x)$, respectively, be the cubic spline FIF and the classical cubic spline with respect to the mesh $\Delta : x_0 < x_1 < \cdots < x_N$, interpolating a set of ordinates $\{y_0, y_1, \ldots, y_N\}$ at the mesh points. Let the cubic polynomial $q_n(\alpha_n, x)$ associated with the IFS for FIF $f_\Delta(x)$ satisfy*

$$\left| \frac{\partial^{1+r} q_n(\tau_n, x)}{\partial \alpha_n \partial x^r} \right| \le K_r \tag{4.2}$$

for $|\tau_n| \in (0, s a_n^r),\ x \in I_n,\ r = 0, 1, 2,$ and $n = 1, 2, \ldots, N$. Then,

$$\|f_\Delta^{(r)} - S_\Delta^{(r)}\|_\infty \le \frac{\|\Delta\|^{2-r} \max_{1 \le n \le N} |\alpha_n|}{|I|^{2-r} - \|\Delta\|^{2-r} \max_{1 \le n \le N} |\alpha_n|} (\|S_\Delta^{(r)}\|_\infty + K_r), \quad r = 0, 1, 2, \tag{4.3}$$

where $|I| = |x_N - x_0|$.

Proof. Denote $\mathcal{B}_r = [-s a_1^r, s a_1^r] \times [-s a_2^r, s a_2^r] \times \cdots \times [-s a_N^r, s a_N^r] \equiv \bigotimes_{n=1}^N [-s a_n^r, s a_n^r]$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathcal{B}_0$ and $r = 0$. Since cubic spline FIF f_Δ is unique for a set of scaling factors $\alpha \in \mathcal{B}_0$ and a prescribed boundary condition, using (3.17) the Read–Bajraktarević operator $T_\alpha^* : \mathcal{F}^* \to \mathcal{F}^*$ can be rewritten as

$$T_\alpha^* f^*(x) = a_n^2 \alpha_n f^*(L_n^{-1}(x)) + a_n^2 q_n(\alpha_n, L_n^{-1}(x)), \quad x \in I_n, n = 1, 2, \ldots, N. \tag{4.4}$$

For a given $\alpha \in \mathcal{B}_0$ and at least one $\alpha_n \ne 0$ in (4.4), cubic spline FIF f_Δ is the fixed point of T_α^*. For $\alpha^* = (0, 0, \ldots, 0) \in \mathcal{B}_0$, the classical cubic spline S_Δ is the fixed point of $T^*_{\alpha^*}$, since in this case $q_n(\alpha_n, x)$ is a polynomial only in x for $n = 1, 2, \dots, N$. Therefore, using (4.4), for $x \in I_n$,

$$\begin{aligned}|T^*_\alpha f_\Delta(x) - T^*_\alpha S_\Delta(x)| &= |a_n^2\alpha_n f_\Delta(L_n^{-1}(x)) + a_n^2 q_n(\alpha_n, L_n^{-1}(x)) \\ &\quad -[a_n^2\alpha_n S_\Delta(L_n^{-1}(x)) + a_n^2 q_n(\alpha_n, L_n^{-1}(x))]| \\ &\le \frac{\|\Delta\|^2}{|I|^2} \max_{1\le n\le N}|\alpha_n| \, \|f_\Delta - S_\Delta\|_\infty.\end{aligned}$$

Since the above inequality holds for $n = 1, 2, \dots, N$, it follows that

$$\|T^*_\alpha f_\Delta - T^*_\alpha S_\Delta\|_\infty \le \frac{\|\Delta\|^2}{|I|^2} \max_{1\le n\le N}|\alpha_n| \, \|f_\Delta - S_\Delta\|_\infty. \tag{4.5}$$

Further, for $x \in I_n$, using (4.4) and Mean Value Theorem,

$$\begin{aligned}|T^*_\alpha S_\Delta(x) - T^*_{\alpha^*} S_\Delta(x)| &= |a_n^2\alpha_n S_\Delta(L_n^{-1}(x)) + a_n^2 q_n(\alpha_n, L_n^{-1}(x)) - a_n^2 q_n(0, L_n^{-1}(x))| \\ &\le a_n^2|\alpha_n|\|S_\Delta\|_\infty + a_n^2|\alpha_n| \left|\frac{\partial q_n(\tau_n, L_n^{-1}(x))}{\partial \alpha_n}\right| \\ &\le \frac{\|\Delta\|^2}{|I|^2} \max_{1\le n\le N}|\alpha_n| \, (\|S_\Delta\|_\infty + K_0).\end{aligned}$$

Since the above inequality holds for $n = 1, 2, \dots, N$,

$$\|T^*_\alpha S_\Delta - T^*_{\alpha^*} S_\Delta\|_\infty \le \frac{\|\Delta\|^2}{|I|^2} \max_{1\le n\le N}|\alpha_n| \, (\|S_\Delta\|_\infty + K_0). \tag{4.6}$$

Using (4.5)–(4.6) together with the inequality

$$\|f_\Delta - S_\Delta\|_\infty = \|T^*_\alpha f_\Delta - T^*_{\alpha^*} S_\Delta\|_\infty \le \|T^*_\alpha f_\Delta - T^*_\alpha S_\Delta\|_\infty + \|T^*_\alpha S_\Delta - T^*_{\alpha^*} S_\Delta\|_\infty$$

gives that

$$\|f_\Delta - S_\Delta\|_\infty \le \frac{\|\Delta\|^2 \max_{1\le n\le N}|\alpha_n|}{|I|^2 - \|\Delta\|^2 \max_{1\le n\le N}|\alpha_n|}(\|S_\Delta\|_\infty + K_0).$$

This proves Lemma 4.1 for $r = 0$. For $r = 1, 2$, the proof of the lemma is analogous to the proof given above for $r = 0$, by taking $\mathcal{B}_1$, $\mathcal{B}_2$, respectively, in place of $\mathcal{B}_0$ and defining Read–Bajraktarević operator on $\mathcal{F}^*_r = \{f \in C^{2-r}(I, \mathbb{R}) \mid f(x_0) = y_0 \text{ and } f(x_N) = y_N\}$ by

$$T^* f^{(r)}(x) = a_n^{2-r} f^{(r)}(L_n^{-1}(x)) + a_n^{2-r} q_n^{(r)}(\alpha_n, L_n^{-1}(x)), \quad r = 1, 2,$$

in place of (4.4). $\square$

For studying the convergence of cubic spline FIFs to a data generating function through sequences of meshes $\{\Delta_k\}$ on $[x_0, x_N]$, define the following types of meshes depending upon vertical scaling factors $\alpha_{k,n}$.

Class A $\{\{\Delta_k\}$: For each $k, \max_{1\le n\le N_k}|\alpha_{k,n}| \le \|\Delta_k\| < 1\}$.

Class B $\{\{\Delta_k\}$: For each $k, |\alpha_{k,i}| > \|\Delta_k\|$ for some $i, 1 \le i \le N_k\}$.

The convergence of a suitable sequence of cubic spline FIFs to the function Φ in $C^2[x_0, x_N]$ generating the interpolation data is described by the following theorem.

THEOREM 4.2. *Let $\Phi \in C^2[x_0, x_N]$ and cubic spline FIFs $f_{\Delta_k}(x)$ satisfy boundary conditions of Type-*I*, Type-*II*, or Type-*III *(periodic) on a sequence of meshes $\{\Delta_k\}$ on $[x_0, x_N]$ with $\lim_{k\to\infty} \|\Delta_k\| = 0$. If $\{\Delta_k\}$ is in Class A, then*

$$\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty = \circ(\|\Delta_k\|^{2-r}), \quad r = 0, 1, 2. \tag{4.7}$$

Further, if $\{\Delta_k\}$ is in Class B, then

$$\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty = \bigcirc(\|\Delta_k\|^{2-r}), \quad r = 0, 1, 2. \tag{4.8}$$

Proof. By Lemma 4.1, each element of the sequence $\{\Delta_k\}$ satisfies

(4.9)
$$\|f^{(r)}_{\Delta_k} - S^{(r)}_{\Delta_k}\|_\infty \le \frac{\|\Delta_k\|^{2-r} \max_{1\le n\le N_k} |\alpha_{k,n}|}{|I|^{2-r} - \|\Delta_k\|^{2-r} \max_{1\le n\le N_k} |\alpha_{k,n}|}(\|S^{(r)}_{\Delta_k}\|_\infty + K_r), \quad r = 0, 1, 2.$$

Further, it is known that [19, 20]

$$\|\Phi^{(r)} - S^{(r)}_{\Delta_k}\|_\infty \le 5\|\Delta_k\|^{2-r}\omega(\Phi^{(r)}; \|\Delta_k\|) \quad (r = 0, 1, 2), \tag{4.10}$$

where $\omega(\Phi; x)$ is the modulus of continuity of $\Phi(x)$. By using the triangle inequality and (4.10), it follows that

$$\|S^{(r)}_{\Delta_k}\|_\infty \le \|\Phi^{(r)}\|_\infty + 5\|\Delta_k\|^{2-r}\omega(\Phi^{(r)}; \|\Delta_k\|). \tag{4.11}$$

The inequality

$$\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty \le \|\Phi^{(r)} - S^{(r)}_{\Delta_k}\|_\infty + \|S^{(r)}_{\Delta_k} - f^{(r)}_{\Delta_k}\|_\infty \tag{4.12}$$

together with (4.9)–(4.11) gives

(4.13)
$$\begin{aligned}\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty \le \|\Delta_k\|^{2-r}\Bigg\{ & 5\omega(\Phi^{(r)}; \|\Delta_k\|) \\ & + \frac{(\|\Phi^{(r)}\|_\infty + 5\|\Delta_k\|^{2-r}\omega(\Phi^{(r)}; \|\Delta_k\|) + K_r) \max_{1\le n\le N_k} |\alpha_{k,n}|}{|I|^{2-r} - \|\Delta_k\|^{2-r} \max_{1\le n\le N_k} |\alpha_{k,n}|}\Bigg\}.\end{aligned}$$

Since $\Phi \in C^2(I)$ and $\max_{1\le n\le N_k} |\alpha_{k,n}| \le \|\Delta_k\| < 1$, the right-hand side of (4.13) tends to zero as $k \to \infty$. The convergence result (4.7) for Class A therefore follows from the error estimate (4.13).

Next, we obtain the convergence result (4.8) for Class B. Since $\max_{1\le n_k\le N_k} |\alpha_{n_k}| \le s < 1$ (cf. definition (4.1)), (4.9) reduces to

$$\|f^{(r)}_{\Delta_k} - S^{(r)}_{\Delta_k}\|_\infty \le \frac{\|\Delta_k\|^{2-r}s}{|I|^{2-r} - \|\Delta_k\|^{2-r}s}(\|S^{(r)}_{\Delta}\|_\infty + K_r), \quad r = 0, 1, 2. \tag{4.14}$$

The inequalities (4.10), (4.11), and (4.14) together with (4.12) give

$$\begin{aligned}\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty \le \|\Delta_k\|^{2-r}\Bigg\{ & 5\omega(\Phi^{(r)}; \|\Delta_k\|) \\ & + \frac{(\|\Phi^{(r)}\|_\infty + 5\|\Delta_k\|^{2-r}\omega(\Phi^{(r)}; \|\Delta_k\|) + K_r)s}{|I|^{2-r} - \|\Delta_k\|^{2-r}s}\Bigg\}.\end{aligned} \tag{4.15}$$

The convergence result (4.8) for Class B now follows from (4.15). ∎

The convergence of a suitable sequence of cubic spline FIFs to the function Φ in $C^3[x_0, x_N]$ generating the interpolation data is given by the following theorem.

THEOREM 4.3. *Let $\Phi \in C^3[x_0, x_N]$ and cubic spline FIFs $f_{\Delta_k}(x)$ satisfy boundary conditions of Type-I, Type-II, or Type-III(periodic) on a sequence of meshes $\{\Delta_k\}$ on $[x_0, x_N]$ with $\lim_{k\to\infty} \|\Delta_k\| = 0$ and $\frac{\|\Delta_k\|}{\min_{1\le n\le N_k} h_{k,n}} \le \beta < \infty$. If $\{\Delta_k\}$ is in Class A, then*

$$\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty = \circ(\|\Delta_k\|^{2-r}), \quad r = 0, 1, 2. \tag{4.16}$$

Further, if $\{\Delta_k\}$ is in Class B, then

$$\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty = \bigcirc(\|\Delta_k\|^{2-r}), \quad r = 0, 1, 2. \tag{4.17}$$

Proof. It is known that [19, 21], for $r = 0, 1, 2$,

$$\|\Phi^{(r)} - S^{(r)}_{\Delta_k}\|_\infty \le \frac{5}{3}\|\Delta_k\|^{3-r}(3 + \bar{K})\omega(\Phi^{(3)}; \|\Delta_k\|), \tag{4.18}$$

where $\bar{K} = 8\beta^2(1 + 2\beta)(1 + 3\beta)$.

Now, (4.9) and (4.18) together with (4.12) give

$$\begin{aligned}\|\Phi^{(r)}-f^{(r)}_{\Delta_k}\|_\infty \le \|\Delta_k\|^{2-r}\Bigg\{&\frac{5}{3}\|\Delta_k\|(3 + \bar{K})\omega(\Phi^{(3)}; \|\Delta_k\|)\\ &+ \frac{(\|\Phi^{(r)}\|_\infty + \frac{5}{3}\|\Delta_k\|(3 + \bar{K})\omega(\Phi^{(3)}; \|\Delta_k\|) + K_r) \max_{1\le n\le N_k} |\alpha_{k,n}|}{|I|^{2-r} - \|\Delta_k\|^{2-r} \max_{1\le n\le N_k} |\alpha_{k,n}|}\Bigg\}.\end{aligned}$$

For the sequence of meshes in Class A or Class B, the relations (4.16)–(4.17) now follow immediately from the above error estimate. ∎

The convergence of a suitable sequence of cubic spline FIFs to the function Φ in $C^4[x_0, x_N]$ generating the interpolation data is described by the following theorem.

THEOREM 4.4. *Let $\Phi \in C^4[x_0, x_N]$ and cubic spline FIFs $f_{\Delta_k}(x)$ satisfy boundary conditions of Type-I or Type-II on a sequence of meshes $\{\Delta_k\}$ on $[x_0, x_N]$ with $\lim_{k\to\infty} \|\Delta_k\| = 0$ and $\frac{\|\Delta_k\|}{\min_{1\le n\le N_k} h_{k,n}} \le \beta < \infty$. If $\{\Delta_k\}$ is in Class A, then*

$$\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty = \circ(\|\Delta_k\|^{2-r}), \quad r = 0, 1, 2. \tag{4.19}$$

Further, if $\{\Delta_k\}$ is in Class B, then

$$\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty = \bigcirc(\|\Delta_k\|^{2-r}), \quad r = 0, 1, 2. \tag{4.20}$$

Proof. It is known that [22]

$$\|\Phi^{(r)} - S^{(r)}_{\Delta_k}\|_\infty \le L_r\|\Phi^{(4)}\|_\infty\|\Delta_k\|^{4-r}, \quad r = 0, 1, 2, 3, \tag{4.21}$$

where $L_0 = 5/384$, $L_1 = 1/24$, $L_2 = 3/8$, and $L_3 = (\beta + \beta^{-1})/2$. The inequalities (4.9) and (4.21) together with (4.12) give the error estimate

(4.22)
$$\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty \leq \|\Delta_k\|^{2-r}\Bigg\{ L_r\|\Phi^{(4)}\|_\infty\|\Delta_k\|^2 + \frac{(\|\Phi^{(r)}\|_\infty + L_r\|\Phi^{(4)}\|_\infty\|\Delta_k\|^{4-r} + K_r)\max_{1\leq n\leq N_k}|\alpha_{k,n}|}{|I|^{2-r} - \|\Delta_k\|^{2-r}\max_{1\leq n\leq N_k}|\alpha_{k,n}|}\Bigg\}.$$

The convergence results (4.19) and (4.20) now follow from (4.22). □

Remarks. 1. Theorem 4.4 generalizes a result of Navascués and Sebastián [23] proved only for uniform meshes with fixed vertical scaling factors.

2. If $\Phi^{(2)}$ satisfies a Hölder condition of order τ, $0 < \tau \leq 1$, Theorem 4.2 gives that, for $r = 0, 1, 2$, $\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty = \circ(\|\Delta_k\|^{2-r})$ if Δ_k is in Class A and $\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty = \bigcirc(\|\Delta_k\|^{2-r})$ if Δ_k is in Class B. This provides an analogue of the corresponding result for classical cubic splines [19, Theorem 2.3.3]. The same estimates on $\|\Phi^{(r)} - f^{(r)}_{\Delta_k}\|_\infty$ follow from Theorem 4.3 or Theorem 4.4 if $\Phi^{(3)}$ or $\Phi^{(4)}$, respectively, satisfies the Hölder condition of order τ, $0 < \tau \leq 1$.

3. It follows from Theorems 4.2–4.4 that the sequence of cubic spline FIFs $f^{(r)}_{\Delta_k}$ converges uniformly to $\Phi^{(r)}$ for $r = 0, 1$ and if Δ_k is in Class A, $f^{(2)}_{\Delta_k}(x)$ converges uniformly to $\Phi^{(2)}(x)$, since, for $r = 2$, the vertical scaling factors can be chosen suitably depending on the mesh norm.

5. Examples of cubic spline FIFs. Using the IFS given by (3.14), we first computationally generate examples of cubic spline FIFs with the set of vertical scaling factors as $\alpha_n = 0.8$, $n = 1, 2, 3$, and the interpolation data as $\{(0,0), (\frac{2}{5}, 1), (\frac{3}{4}, -1), (1, 2)\}$ for the nonperiodic splines and as $\{(0,0), (\frac{2}{5}, 1), (\frac{3}{4}, -1), (1, 0)\}$ for the periodic splines. These interpolation data give $L_1(x) = \frac{2}{5}x$, $L_2(x) = \frac{7}{20}x + \frac{2}{5}$, and $L_3(x) = \frac{1}{4}x + \frac{3}{4}$ in the IFS (3.14) for all our examples of cubic spline FIFs. For constructing an example of the cubic spline FIF with a boundary condition of Type-I, we choose $f'_\Delta(x_0) = 2$ and $f'_\Delta(x_N) = 5$. With these choices, the system of equations (3.11) is solved to get the values of moments M_0, M_1, M_2, M_3 (Table 1). These moments are now used in (3.15) for the construction of $F_n(x, y)$ (Table 2). Iterations of this IFS code generate the desired cubic spline FIF (Figure 2(a)) with a boundary condition of Type-I. Again, to construct an example of the cubic spline FIF with a boundary condition of Type-II, we choose $M_0 = 2$ and M_3 =5. The values of M_1 and M_2 (Table 1) are computed by solving the system (3.12). Using (3.15), the coefficients of $F_n(x, y)$, n =1,2,3, are computed (Table 2). The iterations of the resulting IFS code generate the cubic spline FIF (Figure 2(c)) with a boundary condition of Type-II. An example of the cubic spline FIF with a boundary condition of Type-III (periodic), i.e., $f'_\Delta(x_0) = f'_\Delta(x_3)$ is constructed and $M_0 = M_3$. The values of moments M_0, M_1, M_2, M_3 (Table 1) are computed by solving the system (3.13). The associated IFS code for the periodic cubic spline is obtained from the resulting (3.14). The desired example of the periodic cubic spline FIF (Figure 2(e)) is generated through iterations of this IFS. Similarly, with a 2nd set of vertical scaling factors as $\alpha_1 = \alpha_3 = -0.9$ and $\alpha_2 = 0.9$, the examples of cubic spline FIFs (Figure 2(b), (d), (f)) with boundary conditions of Type-I, Type-II, and Type-III are generated. We note that cubic spline FIFs given by Figure 2(a)–(b) have completely different shapes though they are generated with the same boundary conditions of Type-I, whereas the same boundary conditions give

TABLE 1
Data for cubic spline FIFs with different boundary conditions.

Figures	α_1	α_2	α_3	$f'_\Delta(x_0)$	M_0	M_1	M_2	M_3	$f'_\Delta(x_3)$
2(a)	0.8	0.8	0.8	2	−77.8748	−331.3818	−59.6840	−462.5397	5
2(b)	−0.9	0.9	−0.9	2	26.2835	−31.5521	81.3627	−67.5836	5
2(c)	0.8	0.8	0.8	9.4232	2	−65.0164	93.8441	5	19.4085
2(d)	−0.9	0.9	−0.9	3.4589	2	−34.3620	79.1610	5	13.5633
2(e)	0.8	0.8	0.8	8.1939	5.4523	−43.8970	63.5040	5.4523	8.1939
2(f)	−0.9	0.9	−0.9	4.2258	−3.7995	−30.8481	46.0958	−3.7995	4.2258
2(g)	0.8	0.8	0.8	2	5	−219.5278	25.0565	−281.2847	9.7366
2(h)	−0.9	0.9	−0.9	2	5	−38.5155	79.6443	30.0172	16.9051
2(i)	0.8	0.8	0.8	−49.6	1066.0	111.1	610.8	5	2
2(j)	−0.9	0.9	−0.9	61.5792	−334.8459	59.9983	42.3613	5	2
2(k)	0.8	0.8	0.8	2	129.4060	−44.2624	155.5007	5	17.3112
2(l)	−0.9	0.9	−0.9	2	11.6444	−36.7487	79.9084	5	13.8840
2(m)	0.8	0.8	0.8	−1.4427	2	−297.1132	−11.3357	−423.6607	5
2(n)	−0.9	0.9	−0.9	5.9477	2	−25.6023	78.7498	−61.1447	5
2(o)	0.8	0.8	0.8	9.7621	−14.1432	−79.5646	80.6184	−16.9354	18.9354
2(p)	−0.9	0.9	−0.9	5.7448	−8.1171	−29.6265	77.9665	−9.3573	11.3573

just one interpolating classical cubic spline. Thus, in our approach, an added flexibility is offered to an experimenter depending upon the need of a problem for the choice of a suitable cubic spline FIF. Similarly, Figure 2(c)–(d) gives a comparison of shape and nature of cubic spline FIFs with a boundary condition of Type-II and Figure 2(e)–(f) gives such a comparison for periodic cubic spline FIFs to see the effect of vertical scaling factors on their shapes.

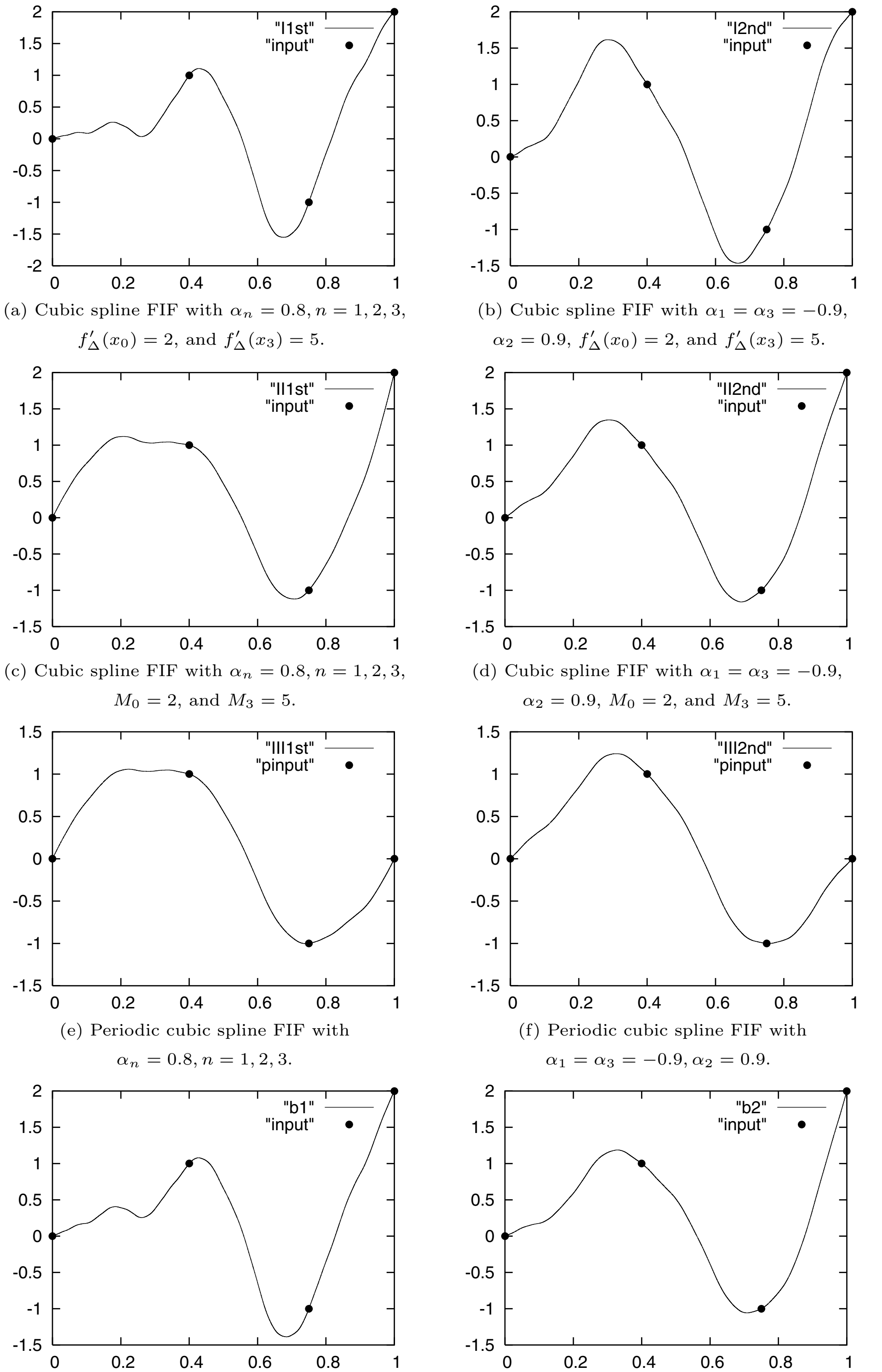
For construction of IFS for cubic spline FIFs (Figure 2(g) and 2(i)) with boundary conditions of Type-IV with first set of vertical scaling factors as $\alpha_n = 0.8, n = 1, 2, 3,$ we choose $f'_\Delta(x_0) = 2$, $M_0 = 5$, and $f'_\Delta(x_3) = 2$, $M_3 = 5$, respectively. The examples of cubic spline FIFs (Figure 2(k) and 2(m)) with boundary conditions of Type-V are constructed with $\alpha_n = 0.8, n = 1, 2, 3,$ by choosing $f'_\Delta(x_0) = 2$, $M_3 = 5$, and $M_0 = 2$, $f'_\Delta(x_3) = 5$, respectively. Finally, for constructing the cubic spline FIF (Figure 2(o)) with a boundary condition of Type-VI, the associated IFS is generated by choosing $\alpha_n = 0.8, n = 1, 2, 3,$ and $f'_\Delta(x_0), M_0, M_3,$ and $f'_\Delta(x_3)$ are chosen such that $3f'_\Delta(x_0) + 2M_0 = 1$ and $f'_\Delta(x_3) + M_3 = 2$. The examples of cubic spline FIFs (Figure 5.1(h), (j), (l), (n), (p)) with boundary conditions of Type-IV, V, or VI are analogously constructed by computing the associated IFS with $\alpha_1 = \alpha_3 = -0.9$ and $\alpha_2 = 0.9$. The effect of vertical scaling factors on the shape and nature of cubic spline FIFs with boundary conditions of Type-IV, V, or VI is demonstrated in Figure 2(g)–(p). Thus, infinitely many cubic spline FIFs with different shapes can be generated by varying scaling factor sets for any prescribed boundary conditions. This gives a vast flexibility in the choice of cubic spline FIF according to the need of the problem.

A normal-size font entry in Table 1 is for the value assumed for a parameter in a particular example. An entry in script-size font in Table 1 is for the value of the parameters that are computed by using (3.10). The entries for the coefficients of $F_n(x, y)$ in Table 2 are computed by using (3.15). All the entries in these tables are rounded off up to four decimal places.

6. Conclusion. A new method is introduced in the present work for the construction of C^r-FIFs so that the complex algebraic method in [1] for construction of C^r-FIFs using complicated matrices is no longer needed. Our method allows admissibility of any kind of boundary conditions while the boundary conditions in [1] are restricted to only at the initial endpoint x_0 of the interval $[x_0, x_N]$. In our approach, r equations involving the spline values or the values of its derivatives at the boundary points are chosen such that the resulting $(r + 2)N + 2r$ equations are linearly independent. This results in generation of a unique C^r-FIF for a prescribed data and a suitable set of vertical scaling factors. This answers a query of Barnsley and Harrington [1, p. 33], regarding uniqueness of the C^r-FIF for a suitable set of vertical scaling

TABLE 2
$\mathbf{F_n(x, y)}$ *for cubic spline FIFs with different boundary conditions.*

Figures	$\mathbf{F_1(x, y)}$	$\mathbf{F_2(x, y)}$	$\mathbf{F_3(x, y)}$
2(a)	$0.128y + 1.446x^3 - 1.246x^2 + 0.544x$	$0.098y + 11.83x^3 - 16.4813x^2 + 2.4552x + 1$	$0.05y - 0.9909x^3 + 0.0817x^2 + 3.8091x - 1$
2(b)	$0.128y - 3.7951x^3 + 3.9951x^2 + 1.088x$	$0.098y + 4.0302x^3 - 3.3814x^2 - 2.8692x + 1$	$0.05y - 2.4315x^3 + 3.2818x^2 + 2.2622x - 1$
2(c)	$0.128y - 2.1981x^3 + 0.1508x^2 + 2.7913x$	$0.098y + 3.0803x^3 - 4.444x^2 - 0.8323x + 1$	$0.05y - 0.8586x^3 + 2.697x^2 + 1.0615x - 1$
2(d)	$0.128y - 0.7661x^3 + 0.5258x^2 + 1.5283x$	$0.098y + 2.1321x^3 - 2.2953x^2 - 2.0572x + 1$	$0.05y - 0.5886x^3 + 2.5711x^2 + 1.13x - 1$
2(e)	$0.128y - 1.3306x^3 + 0.1311x^2 + 2.1995x$	$0.098y + 2.2375x^3 - 3.0902x^2 - 1.1474x + 1$	$0.05y - 0.5819x^3 + 1.7797x^2 - 0.1978x - 1$
2(f)	$0.128y - 1.1279x^3 + 0.6423x^2 + 1.4856x$	$0.098y + 1.7184x^3 - 2.1224x^2 - 1.596x + 1$	$0.05y - 0.595x^3 + 1.5593x^2 + 0.0356x - 1$
2(g)	$0.128y + 1.6313x^3 - 3.5124x^2 + 2.6252x$	$0.098y - 2.0316x^3 + 7.0014x^2 - 7.1903x + 1$	$0.05y + 1.1209x^3 - 3.302x^2 + 5.081x - 1$
2(h)	$0.128y + 4.481x^3 - 5.8543x^2 + 2.6613x$	$0.098y - 2.0316x^3 + 7.0014x^2 - 7.1903x + 1$	$0.05y + 0.383x^3 - 0.1452x^2 + 2.8747x - 1$
2(i)	$0.128y + 15.6608x^3 - 0.793x^2 - 14.1238x$	$0.098y - 30.2834x^3 + 67.7226x^2 - 39.6352x + 1$	$0.05y - 0.2974x^3 + 4.7097x^2 - 1.5124x - 1$
2(j)	$0.128y - 11.0326x^3 + 9.36x^2 + 2.9605x$	$0.098y + 8.4145x^3 - 23.9039x^2 + 13.2688x + 1$	$0.05y - 0.3639x^3 + 3.6069x^2 - 0.1305x - 1$
2(k)	$0.128y - 2.2398x^3 + 2.0705x^2 + 0.9133x$	$0.098y + 5.9094x^3 - 9.052x^2 + 0.9466x + 1$	$0.05y - 0.5053x^3 + 1.6242x^2 + 1.7811x - 1$
2(l)	$0.128y - 1.2367x^3 + 1.7699x^2 + 0.7548x$	$0.098y + 2.3406x^3 - 2.8928x^2 - 1.6683x + 1$	$0.05y - 0.6668x^3 + 2.8246x^2 + 0.9546x - 1$
2(m)	$0.128y + 1.1228x^3 - 0.0231x^2 - 0.3557x$	$0.098y + 12.7309x^3 - 18.1275x^2 + 3.2006x + 1$	$0.05y - 0.7766x^3 - 0.3182x^2 + 3.9947x - 1$
2(n)	$0.128y - 2.4515x^3 + 0.904x^2 + 2.8355x$	$0.098y + 3.3633x^3 - 1.896x^2 - 3.6878x + 1$	$0.05y - 2.0862x^3 + 2.6282x^2 + 2.5705x - 1$
2(o)	$0.128y - 2.1491x^3 + 0.1562x^2 + 2.7369x$	$0.098y - 2.493x^3 - 1.3446x^2 + 1.6416x + 1$	$0.05y + 1.0781x^3 - 2.7304x^2 + 4.5523x - 1$
2(p)	$0.128y + 1.3637x^3 + 0.8732x^2 - 0.9489x$	$0.098y - 1.7662x^3 - 0.8139x^2 + 0.3596x + 1$	$0.05y + 1.7978x^3 - 0.7643x^2 + 2.0789x - 1$



(a) Cubic spline FIF with $\alpha_n = 0.8, n = 1, 2, 3$, $f'_\Delta(x_0) = 2$, and $f'_\Delta(x_3) = 5$.

(b) Cubic spline FIF with $\alpha_1 = \alpha_3 = -0.9$, $\alpha_2 = 0.9$, $f'_\Delta(x_0) = 2$, and $f'_\Delta(x_3) = 5$.

(c) Cubic spline FIF with $\alpha_n = 0.8, n = 1, 2, 3$, $M_0 = 2$, and $M_3 = 5$.

(d) Cubic spline FIF with $\alpha_1 = \alpha_3 = -0.9$, $\alpha_2 = 0.9$, $M_0 = 2$, and $M_3 = 5$.

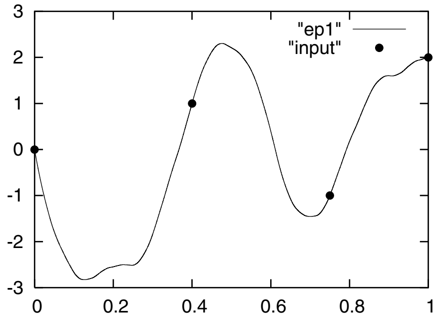
(e) Periodic cubic spline FIF with $\alpha_n = 0.8, n = 1, 2, 3$.

(f) Periodic cubic spline FIF with $\alpha_1 = \alpha_3 = -0.9, \alpha_2 = 0.9$.

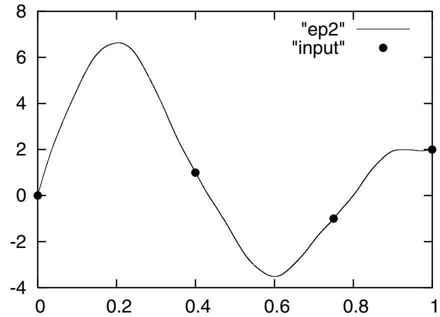
(g) Cubic spline FIF with $\alpha_n = 0.8, n = 1, 2, 3$, $f'_\Delta(x_0) = 2$, and $M_0 = 5$.

(h) Cubic spline FIF with $\alpha_1 = \alpha_3 = -0.9$, $\alpha_2 = 0.9$, $f'_\Delta(x_0) = 2$, and $M_0 = 5$.

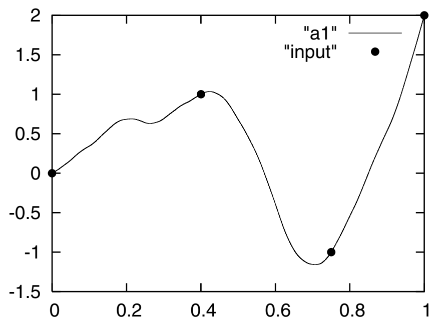
FIG. 2. *Cubic spline FIFs with different boundary conditions.*



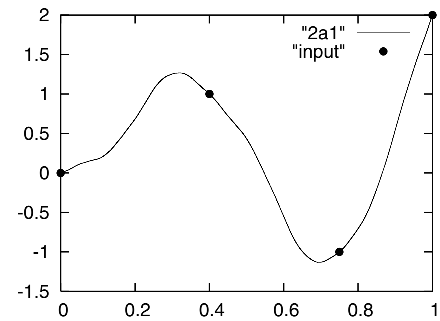
(i) Cubic spline FIF with $\alpha_n = 0.8, n = 1, 2, 3,$ $f'_\Delta(x_3) = 2$, and $M_3 = 5$.



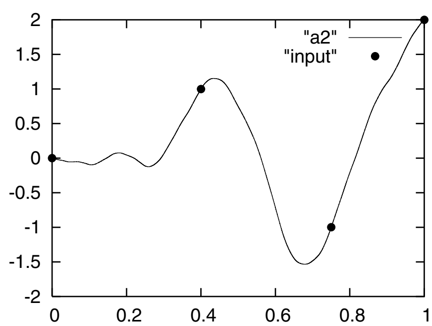
(j) Cubic spline FIF with $\alpha_1 = \alpha_3 = -0.9$, $\alpha_2 = 0.9$, $f'_\Delta(x_3) = 2$, and $M_3 = 5$.



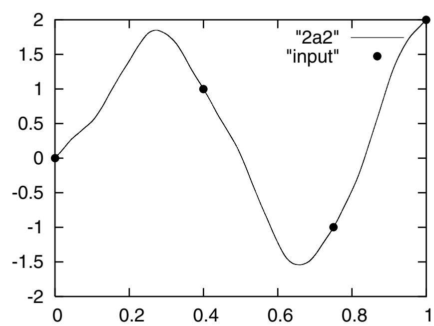
(k) Cubic spline FIF with $\alpha_n = 0.8, n = 1, 2, 3,$ $f'_\Delta(x_0) = 2$, and $M_3 = 5$.



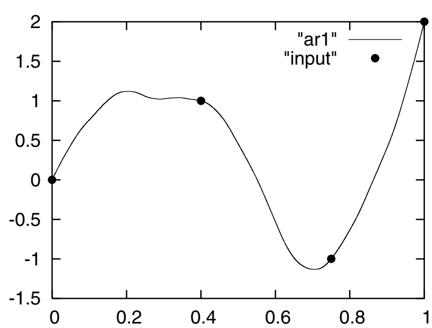
(l) Cubic spline FIF with $\alpha_1 = \alpha_3 = -0.9$, $\alpha_2 = 0.9$, $f'_\Delta(x_0) = 2$, and $M_3 = 5$.



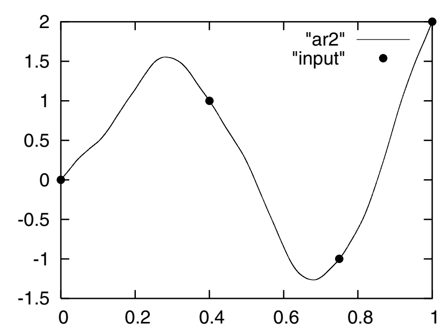
(m) Cubic spline FIF with $\alpha_n = 0.8$, $n = 1, 2, 3$, $f'_\Delta(x_3) = 2$, and $M_0 = 5$.



(n) Cubic spline FIF with $\alpha_1 = \alpha_3 = -0.9$, $\alpha_2 = 0.9$, $f'_\Delta(x_3) = 2$, and $M_0 = 5$.



(o) Cubic spline FIF with $\alpha_n = 0.8, n = 1, 2, 3,$ $3f'_\Delta(x_0) + 2M_0 = 1$, and $f'_\Delta(x_3) + M_3 = 2$.



(p) Cubic spline FIF with $\alpha_1 = \alpha_3 = -0.9$, $\alpha_2 = 0.9, 3f'_\Delta(x_0) + 2M_0 = 1, and\, f'_\Delta(x_3) + M_3 = 2$.

FIG. 2. *Cont.*

factors. The construction of cubic spline FIFs, using the moments $M_n = f''_\Delta(x_n)$, is initiated for the first time in the present work, resulting in a satisfactory generalization of the classical cubic spline theory.

For the data generating function $\Phi \in C^r[x_0, x_n]$, $r = 2, 3$, or 4, it is proved that (cf. Theorems 4.2–4.4), the sequence of cubic spline FIFs $\{f_{\Delta_k}\}$ converges to Φ with arbitrary degree of accuracy for the sequences of meshes in Class A or Class B for boundary conditions of Type-I, Type-II, or Type-III. Our convergence results in section 4 are obtained with more general conditions than those in [23] wherein only uniform meshes are considered in the case $\Phi \in C^{(4)}[x_0, x_n]$. The upper bounds on error in approximation of Φ and its derivatives by cubic spline FIFs f_Δ and its derivatives, respectively, with different boundary conditions are also obtained by results in section 4. As a consequence of our results, the data generating function Φ that satisfies $\Phi^{(2)} \in Lip\ \tau$, $0 < \tau < 1$, can be approximated satisfactorily by a fractal function f_Δ by choosing vertical scaling factors suitably such that $f_\Delta^{(2)} \in Lip\ \tau$.

The vertical scaling factors α_n are important parameters in the construction of C^r-FIFs or cubic spline FIFs. For given boundary conditions, in our approach an infinite number of C^r-FIFs or cubic spline FIFs can be constructed interpolating the same data by choosing different sets of vertical scaling factors. Thus, according to the need of an experiment for simulating objects with smooth geometrical shapes, a large flexibility in the choice of a suitable interpolating smooth fractal spline is offered by our approach. As in the case of vast applications of classical splines in CAM, CAGD, and other mathematical, engineering applications [12, 13, 14, 15], it is felt that cubic spline FIFs generated in the present work can find rich applications in some of these areas. Since the cubic spline FIF is invariant in all scales, it can also be applied to image compression and zooming problems in image processing. Further, as classical cubic splines are a special case of cubic spline FIFs, it should be possible to use cubic spline FIFs for mathematical and engineering problems where the classical spline interpolation approach does not work satisfactorily.

Acknowledgment. The authors are thankful to Dr. Ian Sloan for his valuable suggestions.

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