Assignment 10: Functions of several variables (Continuity and Differentiabilty)

1. **(D)** Examine the following functions for continuity at the point $(0,0)$, where $f(0,0) = 0$ and $f(x,y)$ for $(x,y) \neq (0,0)$ is given by

   $i)$ $\mid x \mid + \mid y \mid$ $i)$ $\frac{xy}{\sqrt{x^2+y^2}}$ $ii)$ $\frac{xy}{x^2+y^2}$ $iii)$ $\frac{x^4-y^2}{x^4+y^2}$ $iv)$ $\frac{x^2y}{x^4+y^2}$.

4. **(D)** Let $f(x,y)$ be defined in $S = \{(x,y) \in \mathbb{R}^2 : a < x < b,\ c < y < d\}$. Suppose that the partial derivatives of f exist and are bounded in S. Then show that f is continuous in S.

5. **(D)** Let $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and 0, otherwise. Prove that

   (a) $f_x(0,y) = -y$ and $f_y(x,0) = x$ for all x and y;

   (b) $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ and (c) $f(x,y)$ is differentiable at $(0,0)$.

Assignment 10 - Solutions

1. (a) The function is continuous everywhere as $|x_n| + |y_n| \to |x_0| + |y_0|$ whenever $(x_n, y_n) \to (x_0, y_0)$.

   (b) The function is continuous because $\left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \frac{|x^2+y^2|}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} \to 0$, as $(x, y) \to 0$.

   (c) Let $y = mx$. Then $\frac{xy}{x^2+y^2} = \frac{m}{1+m^2}$ $\forall (x,y) \neq (0,0)$. Therefore, f is discontinuous at $(0,0)$.

   (d) Let $y = mx^2$. Then $\frac{x^4-y^2}{x^4+y^2} = \frac{1-m^2}{1+m^2}$ $\forall (x,y) \neq (0,0)$. Therefore, f is discontinuous at $(0,0)$.

   (e) Let $y = mx^2$. Then $\frac{x^2y}{x^4+y^2} = \frac{m}{1+m^2}$ $\forall (x,y) \neq (0,0)$. Therefore, f is discontinuous at $(0,0)$.

4. Let $|f_x(x,y)| \leq M$ and $|f_y(x,y)| \leq M$, $\forall (x,y) \in S$. Then,

   $f(x+h, y+k) - f(x,y) = f(x+h, y+k) - f(x+h, y) + f(x+h, y) - f(x,y) = k f_y(x+h, y+\theta_1 k) + h f_x(x+\theta_2 h, y)$, (by the mean value theorem).

   Hence, $|f(x+h, y+k) - f(x,y)| \leq M(|h| + |k|) \leq 2M\sqrt{h^2+k^2}$.

   Hence, for $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2M}$ or use the sequential argument to show that the function is continuous.

5. (a) Note that $f_y(h,0) = \lim_{k \to 0} \frac{f(h,k)-f(h,0)}{k} = \lim_{k \to 0} \frac{h(h^2-k^2)}{h^2+k^2} = h$.

   (b) Note that $f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = 1$. Similarly, $f_{xy}(0,0) = -1$.

   (c) Done in the class (see the lecture notes).