Assignment 11: Directional derivatives, Maxima, Minima, Lagrange Multipliers

1. **(D)** Let $f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2}$ if $y \neq 0$ and $f(x,y) = 0$ if $y = 0$. Show that f is continuous at $(0,0)$, it has all directional derivatives at $(0,0)$ but it is not differentiable at $(0,0)$.

2. **(T)** Let $f(x,y) = \frac{1}{2}\Big(\big|\,|x| - |y|\,\big| - |x| - |y|\Big)$. Is f continuous at $(0,0)$? Which directional derivatives of f exist at $(0,0)$? Is f differentiable at $(0,0)$?

3. **(T)** Find the equation of the surface generated by the normals to the surface $x + 2yz + xyz^2 = 0$ at all points on the z-axis.

4. **(T)** Examine the following functions for local maxima, local minima and saddle points:

   $i)$ $4xy - x^4 - y^4$ $\qquad ii)$ $x^3 - 3xy$

5. **(D)** Let $f(x,y) = 3x^4 - 4x^2y + y^2$. Show that f has a local minimum at $(0,0)$ along every line through $(0,0)$. Does f have a minimum at $(0,0)$? Is $(0,0)$ a saddle point for f?

6. **(T)** Find the absolute maxima of $f(x,y) = xy$ on the unit disc $\{(x,y) : x^2 + y^2 \leq 1\}$.

7. **(D)** Assume that among all rectangular boxes with fixed surface area of 20 square meters, there is a box of largest possible volume. Find its dimensions.

8. **(D)** L&T produces steel boxes at three different plants in amounts x, y and z, respectively, producing an annual revenue of $R(x,y,z) = 8xyz^2 - 200(x+y+z)$. The company is to produce 100 units annually. How should production be distributed to maximize revenue?

9. **(T)** Minimize the quantity $x^2+y^2+z^2$ subject to the constraints $x+2y+3z = 6$ and $x + 3y + 9z = 9$.

Assignment 11 - Solutions

1. Note that $\mid f(x,y) - f(0,0) \mid = \sqrt{x^2+y^2}$. Hence the function is continuous.

   For $\| (u_1,u_2) \| = 1$, $\lim_{t\to 0} \frac{f(tu_1,tu_2)}{t} = 0$ if $u_2 = 0$ and $\frac{u_2}{|u_2|}$ if $u_2 \neq 0$. Therefore directional derivatives in all directions exist.

   Note that $f_x(0,0) = 0$ and $f_y(0,0) = 1$. If f is differentiable at $(0,0)$ then $f'(0,0) = \alpha = (0,1)$. Note that

   $$\epsilon(h,k) = \frac{\frac{k}{|k|}\sqrt{h^2+k^2} - k}{\sqrt{h^2+k^2}} \nrightarrow 0 \text{ as } (h,k) \to (0,0).$$

   For example, $h = k$ gives $(\sqrt{2}-1)\frac{k}{|k|} \nrightarrow 0$ as $k \to 0$. Therefore the function is not differentiable at $(0,0)$.

2. $|f(x,y) - f(0,0)| \leq |x| + |y|$. Thus f is continuous at $(0,0)$.

   $\lim\limits_{t\to 0} \dfrac{f(tu_1,tu_2) - f(0,0)}{t} = \dfrac{|t|}{2t}\{| \, |u_1| - |u_2| \, | - |u_1| - |u_2|\}$.

   Hence, the directional derivatives of f exist at $(0,0)$ if and only if $| \, |u_1| - |u_2| \, | = |u_1| + |u_2|$, that is, either $u_1 = 0$ or $u_2 = 0$. Since the directional derivatives in all direction do not exist, the function cannot be differentiable at $(0,0)$.

3. The normal to the given surface is $N = (1 + yz^2, 2z + xz^2, 2y + 2xyz)$. The normal at a point on the z-axis is $(1, 2t, 0)$. If (x,y,z) is any point on the given surface generated then $\dfrac{x}{1} = \dfrac{y}{2t}$, $z = t$. Hence, the surface generated is $y = 2xz$ (by eliminating t).

4. (i) For $f(x,y) = 4xy - x^4 - y^4, f_x(x_0,y_0) = f_x(x_0,y_0) = 0$ for $(x_0,y_0) = (0,0),(1,1)$ or $(-1,-1)$. These are the critical points. By second derivative test, $(0,0)$ ia a saddle point and $(-1,1)$ and $(1,1)$ are local maxima.

   (ii) $f(x,y) = x^3 - 3xy^2, f_x(x_0,y_0) = f_x(x_0,y_0) = 0$ for $(x_0,y_0) = (0,0)$. So $(0,0)$ is the only critical point . Second derivative fails here. Along $y = 0, f(x,y) = x^3$, hence $(0,0)$ is a saddle point.

5. Let $f(x,y) = 3x^4 - 4x^2y + y^2$. Along, the x-axis, the local minimum of the function is at $(0,0)$. Let $x = r\cos\theta$ and $y = r\sin\theta$, for a fixed $\theta \neq 0, \pi$ (or let $y = mx$). Then,

   $$f(r\cos\theta, r\sin\theta) = 3r^4\sin^4\theta - 4r^3\cos^2\theta\sin\theta + r^2\sin^2\theta$$

   which is function of one variable. By the second derivative test (for functions of one variable), we see that $(0,0)$ is a local minima.

   Since, $f(x,y) = (3x^2 - y)(x^2 - y)$, we see that in the region between the parabolas $3x^2 = y$ and $y = x^2$, the function takes negative values and is positive everywhere else. Thus, $(0,0)$ is a saddle point for f.

6. $f(x,y) = xy \Rightarrow f_x = y, f_y = x$. Clearly, $(0,0)$ is the only critical point. $f(0,0) = 0$.

   Let us use the method of lagrange multipliers on $x^2 + y^2 = 1$. Consider the function $F(x,y,z) = xy - \lambda(x^2 + y^2 - 1)$. Here, $F_x = y - 2\lambda x$, $F_y = x - 2\lambda y$ and $F_\lambda = x^2 + y^2 - 1$. Therefore, $y = 2\lambda x$, $x = 2\lambda y \Rightarrow x = 0 \iff y = 0$. But, $x^2 + y^2 = 1$. Hence, $y = 4\lambda^2 y$.

$\lambda = \pm\frac{1}{2}$ and $y = \pm x \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ and $x = \pm\frac{1}{\sqrt{2}}$.

Hence, we need to compute the absolute maximum and minimum at the points $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. The absolute maximum is attained at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

7. Let the box have sides of length x, y, $z > 0$. Then $V(x, y, z) = xyz$ and $xy+yz+xz = 10$. Using the method of lagrange multipliers, we see that $yz = \lambda(y+z)$, $xz = \lambda(x+z)$ and $xy = \lambda(x+y)$. It is easy to see that $x, y, z > 0$. Now, we can see that $x = y = z$ and therefore, $x = y = z = \sqrt{\frac{10}{3}}$.

8. Use the method of lagrange multipliers, where $\bigtriangledown R = \lambda \bigtriangledown F$. Here, $F(x, y, z) = x + y + z = 100$.

9. Let $F(x, y, z) = x^2 + y^2 + z^2$, $g(x, y, z) = x + 2y + 3z$ and $h(x, y, z) = x + 3y + 9z$, where $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

   Let λ and μ be such that $\bigtriangledown F = \lambda \bigtriangledown h + \mu \bigtriangledown g$.

   We get
   $$\lambda + \mu = 2x,\ 2\lambda + 3\mu = 2y \text{ and } 3\lambda + 9\mu = 2z \qquad (1).$$

   From here, using $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$, we get $7\lambda + 17\mu = 6$ and $34\lambda + 91\mu = 18$.

   Hence, $\mu = -\dfrac{78}{59}$ and $\lambda = \dfrac{240}{59}$.

   From equation (1), we get $2(x^2 + y^2 + z^2) = 6\lambda + 9\mu$, hence the minimum value of of f is $\frac{369}{59}$.