# Assignment 12 : Double Integrals

1. **(D)** Evaluate the integral $\iint\limits_R (x+y)^2 dxdy$ over the triangle $R$ with vertices $(0,0),\ (2,2)$ and $(0,1)$.

2. **(T)** Evaluate the following integrals:

   $i)\ \int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} \sqrt{1-y^2}dydx \qquad ii)\ \int\limits_0^\pi \int\limits_x^\pi \frac{\sin y}{y} dydx \qquad iii)\ \int\limits_0^1 \int\limits_y^1 x^2 \exp^{xy} dxdy.$

3. **(D)** Evaluate the integral $\iint\limits_R (x^2 - y^2)dxdy$ over the square $R$ with vertices $(0,0),\ (1,-1),\ (2,0)$ and $(1,1)$.

4. **(T)** Evaluate $\iint\limits_R xdxdy$ where $R$ is the region $1 \leq x(1-y) \leq 2$ and $1 \leq xy \leq 2$.

5. **(D)** The cylinder $x^2 + z^2 = 1$ is cut by the planes $y = 0, z = 0$ and $x = y$. Find the volume of the region in the first octant.

6. **(T)** Compute $\lim\limits_{a \longrightarrow \infty} \int\int\limits_{D(a)} \exp^{-(x^2+y^2)} dxdy$, where

   $i)$ $D(a) = \{(x,y) : x^2 + y^2 \leq a^2\}$ and $ii)$ $D(a) = \{(x,y) : 0 \leq x \leq a,\ 0 \leq y \leq a\}$.

   Hence prove that $\int\limits_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

7. **(D)** Change the order of integration to prove that

   $i)\ \int\limits_0^x \int\limits_0^u e^{m(x-t)} f(t)dtdu = \int\limits_0^x (x-t)e^{m(x-t)} f(t)dt,$

   $ii)\ \int\limits_0^x \int\limits_0^v \int\limits_0^u e^{m(x-t)} f(t)dtdudv = \int\limits_0^x \frac{(x-t)^2}{2} e^{m(x-t)} f(t)dt.$

# Assignment 12 - Solutions

1. $\iint_R (x+y)^2 = \int_0^2 \int_x^{\frac{x}{2}+1} (x+y)^2\, dx\, dy = \frac{1}{3}\int_0^2 [(\frac{3x}{2}+1)^3 - (2x)^3]dx = \frac{7}{2}.$

2. (a) $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\, dy\, dx = \int_0^1 (\int_0^{\sqrt{1-y^2}} \sqrt{1-y^2} dx) dy = \int_0^1 (1-y^2) dy = \frac{2}{3}.$

   (b) $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y} dy dx = \int_0^{\pi}\int_0^{y} \frac{\sin y}{y} dx dy = \int_0^{y} \sin y dy = 2.$

   (c) $\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy = \int_0^1 \int_0^x x^2 e^{xy}\, dy\, dx = \int_0^1 x(e^{x^2}-1)dx = \frac{e-2}{2}.$

3. Let $x+y=u$ and $x-y=v$. Then $0 \le u, v \le 2$. Now, since $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$, we have the Jacobian determinant,

$$J = \begin{vmatrix} \frac{\delta x}{\delta u} & \frac{\delta x}{\delta v} \\ \frac{\delta y}{\delta u} & \frac{\delta y}{\delta v} \end{vmatrix} = \frac{1}{2}.$$

   Thus, $\iint_R (x^2-y^2) dxdy = \int_0^2 \int_0^2 uv \frac{dudv}{2} = 2.$

   (This problem can also be solved directly by using the Fubini's theorem.)

4. Choose $u = x(1-y)$ and $v = xy$. Then, $1 \le u \le 2$ and $1 \le v \le 2$. Note that $x = u+v$, $y = \frac{v}{u+v}$ and $\mid J(u,v) \mid = \frac{1}{u+v}$. Therefore, $\iint_R x dxdy = \int_1^2 \int_1^2 dvdu = 1$

5. Note that the projection $R$ of the surface in the xy-plane is bounded by $x=0, x=1, y=0$ and $y=x$. The surface is defined by the function $z = f(x,y) = \sqrt{1-x^2}$. Therefore,

$$V = \iint_R z dxdy = \int_0^1 (\int_0^x \sqrt{1-x^2} dy) dx = \frac{1}{3}$$

6. (a) Let $D(a) = \{(x,y) : x^2+y^2 \le a\}$. Then

$$\iint_{D(a)} e^{-(x^2+y^2)} dxdy = \int_0^{2\pi}\int_0^{a} e^{-r^2} r dr d\theta = \pi(1-e^{-a^2}).$$

   Therefore, $\lim_{a\to\infty} \iint_{D(a)} e^{-(x^2+y^2)} dxdy = \pi.$

(b) Let $D_1(a) = \{(x,y) : x, y \geq 0,\ x^2+y^2 \leq a\}$ and $D_2(a) = \{(x,y) : 0 \leq x, y \leq a\}$. Note that

$$\iint\limits_{D_1(a)} e^{-(x^2+y^2)}dxdy \leq \iint\limits_{D_2(a)} e^{-(x^2+y^2)}dxdy \leq \int\int\limits_{D_1(\sqrt{2}a)} e^{-(x^2+y^2)}dxdy.$$

Now, use the sandwich theorem. We see that

$$\lim_{a\to\infty} \iint\limits_{D_2(a)} e^{-(x^2+y^2)}dxdy = \lim_{a\to\infty} \iint\limits_{D_1(a)} e^{-(x^2+y^2)}dxdy = \frac{1}{4}\pi.$$

Moreover,

$$(\int_0^\infty e^{-x^2}dx)^2 = (\int_0^\infty e^{-x^2}dx)(\int_0^\infty e^{-y^2}dy) = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy = \frac{\pi}{4}.$$

7. (a) $$\int_0^x\int_0^u e^{m(x-t)}f(t)dtdu = \int_0^x\int_t^x e^{m(x-t)}f(t)dudt = \int_0^x (x-t)e^{m(x-t)}f(t)dt.$$

(b) $$\int_0^x(\int_0^v\int_0^u e^{m(x-t)}f(t)dtdu)dv = \int_0^x\int_0^v\int_t^v e^{m(x-t)}f(t)dudtdv$$

$$= \int_0^x\int_0^v e^{m(x-t)}f(t)(v-t)dtdv = \int_0^x\int_t^x e^{m(x-t)}f(t)(v-t)dvdt$$

$$= \int_0^x \frac{(x-t)^2}{2}e^{m(x-t)}f(t)dt.$$