Assignment 13 : Triple Integrals, Surface Integrals, Line integrals

1. **(D)** Integrate $ze^{x^2+y^2}dxdydz$ over the cylinder $x^2 + y^2 \leq 4,\ 2 \leq z \leq 3$.

2. **(T)** Evaluate the integral $\iiint\limits_{W} \frac{dzdydx}{\sqrt{1+x^2+y^2+z^2}}$; where W is the ball $x^2+y^2+z^2 \leq 1$.

3. **(D)** Find the area of the surface of the portion of the sphere $x^2 + y^2 + z^2 = 4a^2$ that lies inside the cylinder $x^2 + y^2 = 2ax$.

4. **(T)** What is the integral of the function x^2z taken over the entire surface of a right circular cylinder of height h which stands on the circle $x^2 + y^2 = a^2$. What is the integral of the given function taken throughout the volume of the cylinder.

5. **(D)** Compute $\iint\limits_{S} xyd\sigma$, where S is the surface of the cone $x = r\cos t,\ y = r\sin t, z = r$ for $0 \leq r \leq 1$ and $0 \leq t \leq 2\pi$.

6. **(T)** Find the line integral of the vector field $F(x, y, z) = y\vec{i} - x\vec{j} + \vec{k}$ along the path $\mathbf{c}(t) = (\cos t, \sin t, \frac{t}{2\pi})$
$0 \leq t \leq 2\pi$ joining $(1, 0, 0)$ to $(1, 0, 1)$.

7. **(D)** Evaluate $\int\limits_{C} \frac{-ydx+xdy}{x^2+y^2}$, where $C := \{(x, y) : x^2 + y^2 = 1\}$.

8. **(T)** Evaluate $\int\limits_{C} T \cdot dR$, where C is the circle $x^2 + y^2 = 1$ and T is the unit tangent vector.

9. **(T)** Show that the integral $\int\limits_{C} yzdx + (xz + 1)dy + xydz$ is independent of the path C joining $(1, 0, 0)$ and $(2, 1, 4)$.

Assignment 13 - Solutions

1. Use cylindrical coordinates. Let $x = r\cos\theta$, $x = r\sin\theta$ and $z = z$. Then, $x^2 + y^2 \leq 4,\ 2 \leq z \leq 3 \Longrightarrow 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 2,\ 2 \leq z \leq 3$.

   Hence, $\displaystyle\int_2^3\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} ze^{x^2+y^2}dydxdz = \int_2^3\int_0^2\int_0^{2\pi} ze^{r^2}rdrd\theta dz = \frac{5\pi}{2}(e^4 - 1).$

2. Use spherical coordinates. Let $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$, where $0 \leq \rho \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

$$\iiint_W \frac{dzdydx}{\sqrt{1+x^2+y^2+z^2}} = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \frac{\rho^2\sin\phi d\rho d\theta d\phi}{\sqrt{1+\rho^2}} = 2\pi(\sqrt{2} - \ln(1+\sqrt{2})).$$

3. Done in the class.

   The surface area $= 2\iint_R \sqrt{1+f_x^2+f_y^2}dxdy$ where R is the circular disk with the boundary : $x^2+y^2 = 2ax$.

   The surface area $= 4\int_0^{\frac{\pi}{2}}\int_0^{2acos\theta} \frac{2ardrd\theta}{\sqrt{4a^2-r^2}}$.

4. In the cylinder there are three surfaces S_1, S_2 and S_3 where

   (a) S_1 : The base of the cylinder, i.e., $z = 0$,

   (b) S_2 : The top of the cylinder i.e., $z = h$,

   (c) S_3 : The curved surface of the cylinder.

   (a) On S_1, the integral is zero.

   (b) The surface integral over $S_2 = \displaystyle\iint_{S_2} x^2zd\sigma = \int_0^a\int_0^{2\pi}(r\cos\theta)^2hrd\theta dr = \frac{ha^4\pi}{4}.$

   (c) A parametric representation of S_3 is

$$r(u,v) = (a\cos u, a\sin u, v), 0 \leq u \leq 2\pi, 0 \leq v \leq h.$$

   The surface integral over $S_3 = \iint_{S_3} x^2zd\sigma = \int_0^h\int_0^{2\pi} x^2z \parallel r_u \times r_v \parallel dudv$

   $= \int_0^h\int_0^{2\pi}(a\cos u)^2v\sqrt{EG-F^2}dudv$, where $E = r_u \cdot r_u,\ G = r_v \cdot r_v$ and $F = r_u \cdot r_v$.

   Note that $\sqrt{EG-F^2} = a$. Therefore, $\iint_{S_3} x^2zd\sigma = \frac{a^3h^2\pi}{2}$.

   Hence, the required integral is $\frac{ha^4\pi}{4} + \frac{a^3h^2\pi}{2}$.

   Over the entire volume, the integral is

$$V = \int_0^h\int_0^{2\pi}\int_0^a (r\cos\theta)^2zrdrd\theta dz = \frac{h^2\pi a^4}{8}.$$

5. $\phi(r,t) = (r\cos t, r\sin t, t)$, $0 \le r \le 1$ and $0 \le \theta \le 2\pi$ is the parametric equation of the cone.

   $\phi_r = (\cos\theta, \sin\theta, 1)$ and $\phi_\theta = (-r\sin\theta, r\cos\theta, 0)$.

   $\phi_r.\phi_r = 2$, $\phi_\theta.\phi_\theta = r^2$ and $\phi_r.\phi_\theta = 0$.

   Hence the required surface integral is

$$\int_0^1 \int_0^{2\pi} r^2 \sin\theta\cos\theta \sqrt{(\phi_r.\phi_r).(\phi_\theta.\phi_\theta) - (\phi_r.\phi_\theta)^2}\, dr d\theta = \frac{\sqrt{2}}{4} \int_0^{2\pi} \sin\theta\cos\theta d\theta = 0.$$

6. $\int_C (y, -x, 1) \cdot dR = \int_0^{2\pi} ((\sin t)(-\sin t)dt - \cos t\cos t + \frac{1}{2\pi})dt.$

7. Let us consider $C = (cost, sint), 0 \le t \le 2\pi$. Then $\int_C \frac{-ydx+xdy}{x^2+y^2} = \int_0^{2\pi} \frac{sin^2t+cos^2t}{sin^2t+cos^2t} dt = 2\pi.$

8. Take $C = R(t) = (cost, sint),\ 0 \le t \le 2\pi$. Then

$$\int_C T \cdot dR = \int_0^{2\pi} T(t) \cdot R'(t)dt = \int_0^{2\pi} \frac{R'(t)}{\| R'(t) \|} \cdot R'(t)dt = 2\pi$$

9. If $F = yzi + (xz+1)j + xyk$, then $F = \nabla\varphi$, where $\varphi(x,y,z) = xyz + y$. Hence, by the 2nd fundamental theorem of calculus for line integrals, the problem follows.