Assignment 14 : Green's /Stoke's /Gauss's Theorems

1. **(T)** Use Green's Theorem to compute $\int_C (2x^2 - y^2)\, dx + (x^2 + y^2)\, dy$ where C is the boundary of the region $\{(x, y) : x, y \geq 0 \;\; \& \;\; x^2 + y^2 \leq 1\}$.

2. **(D)** Show that the value of the line integral $\int xy^2 dx + (x^2 y + 2x)dy$ around any square depends only on the size of the square and not on its location in the plane.

3. **(D)** Evaluate $\int_C \frac{xdy - ydx}{x^2+y^2}$ along any simple closed curve in the xy plane not passing through the origin. Distinguish the cases where the region R enclosed by C:
   (a) includes the origin (b) does not include the origin.

4. **(T)** Use Stoke's Theorem to evaluate the line integral $\int_C -y^3\, dx + x^3\, dy - z^3\, dz$, where C is the intersection of the cylinder $x^2+y^2 = 1$ and the plane $x+y+z = 1$ and the orientation of C corresponds to counterclockwise motion in the xy-plane.

5. **(D)** Verify the Stoke's Theorem where $\overrightarrow{F} = (y, z, x)$ and S is the part of the cylinder $x^2 + y^2 = 1$ cut off by the planes $z = 0$ and $z = x + 2$, oriented with $\overrightarrow{n}$ pointing outward.

6. **(T)** Let $\overrightarrow{F} = \frac{\overrightarrow{r}}{|\overrightarrow{r}|^3}$ where $\overrightarrow{r} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$ and let S be any surface that surrounds the origin. Prove that $\iint_S \overrightarrow{F}.n\; d\sigma = 4\pi$.

7. **(T)** Let D be the domain inside the cylinder $x^2 + y^2 = 1$ cut off by the planes $z = 0$ and $z = x + 2$. If $\vec{F} = (x^2 + ye^z, y^2 + ze^x, z + xe^y)$, use the divergence theorem to evaluate $\iint_{\partial D} F \cdot \mathbf{n}\; d\sigma$.

Assignment 14 - Solutions

1. $M = 2x^2 - y^2$ and $N = x^2 + y^2$. By Green's Theorem

$\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy = \int_0^1 \int_0^{\sqrt{1-x^2}} (N_x - M_y)dy\,dx$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} 2(x+y)dy\,dx = \frac{4}{3}$.

2. Let R be a square with the boundary C. Then by Green's theorem $\int_C xy^2dx + (x^2y + 2x)dy = \iint_R 2dxdy = 2$ Area(R).

3. (a) Let $M = -\dfrac{y}{x^2+y^2}$ and $N = \dfrac{x}{x^2+y^2}$. Suppose $(0,0) \in R$.

   Since the function is not defined at $(0,0)$, choose C_α to be a circle of radius α containing $(0,0)$ and C lies in the interior of R. Let D be the region bounded by the simple closed curves C and C_α. In this region $N_x - M_y = 0$.

   By Green's Theorem, $\int_{C \cup C_\alpha} Mdx + Ndy = \iint_D (N_x - M_y)dx\,dy = 0$.

   Hence, $\int_C Mdx + Ndy = \int_{-C_\alpha} Mdx + Ndy = 2\pi$.

   (b) For a simple closed curve C not containing $(0,0)$, by Green's theorem, we have $\int_C Mdx + Ndy = 0$.

4. Let $F = -y^3\vec{i} + x^3\vec{j} - z^3\vec{k}$. By Stoke's Theorem, $\int_{\partial S} F.dr = \int\int_S (curl\, F).\vec{n}d\sigma$.

   Note that $\nabla \times F = 3(x^2+y^2)\vec{k}$. Hence, $\int_{\partial S} F.dr = \iint_D 3(x^2+y^2)dxdy = \frac{3\pi}{2}$.

5. Let us first evaluate $\iint_S curlF \cdot nd\sigma$. Consider $S := r(\theta, z)$ where

$$r(\theta, z) = (\cos\theta, \sin\theta, z) \quad \text{where} \quad 0 \le \theta \le 2\pi \quad \text{and} \quad 0 \le z \le 2 + \cos\theta.$$

Note that

$$curlF = -i - j - k,\ n = \frac{r_\theta \times r_z}{\| r_\theta \times r_z \|} = \cos\theta i + \sin\theta j + 0k \text{ and } \sqrt{EG - F^2} = 1.$$

Therefore,

$$\iint_S curlF \cdot nd\sigma = \int_{-\pi}^{\pi} \int_0^{2+\cos\theta} (-\cos\theta - \sin\theta)dzd\theta = -\pi$$

Let C_1 and C_2 be the boundary curves of the surface S which are lying in the plane $z = 0$ and $z = x + 2$ respectively. Consider the parameterizations

$$C_1 := R(\theta) = \cos\theta i + \sin\theta j, \quad 0 \le \theta \le 2\pi$$

and

$$C_2 := R(\theta) = \cos\theta i + \sin\theta j + (2+\cos\theta)k, \quad 0 \le \theta \le 2\pi.$$

Then

$$\oint_{C_1} F \cdot dR = \int_0^{2\pi} -\sin^2\theta d\theta = -\pi$$

and

$$\oint_{C_2} F \cdot dR = \int_{2\pi}^{0} \ldots\ldots \quad = 0$$

(note that the direction of the integration over C_2 is in the clockwise direction(see the figure))).

6. Note that $div\ F = 0$. By divergence theorem

$$\iint_S F \cdot n d\sigma = \iint_{S_\rho} F \cdot n d\sigma$$

where S_ρ is a sphere of (small) radius ρ with center at origin. On S_ρ, $n = \frac{1}{\rho}(xi+yj+zk)$ and hence $F \cdot n = \frac{1}{\rho^2}$. Therefore,

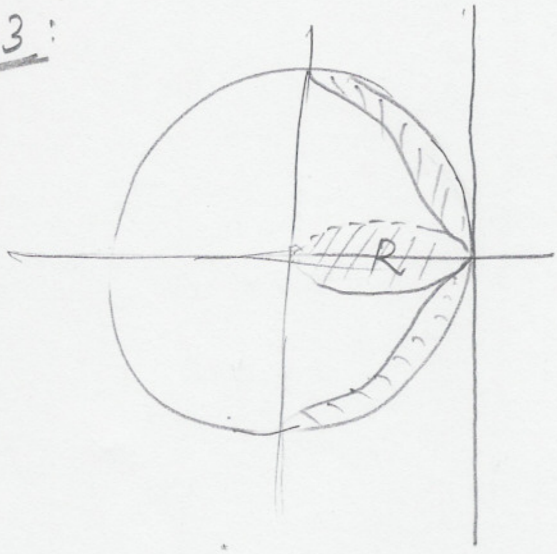
$$\iint_{S_\rho} F \cdot n d\sigma = \frac{1}{\rho^2}\iint_{S_\rho} d\sigma = \frac{1}{\rho^2}4\pi\rho^2 = 4\pi.$$

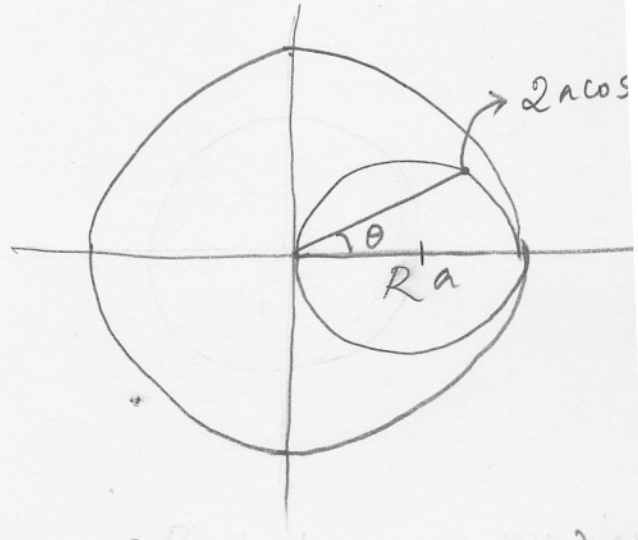
7. $div\,F = 2x + 2y + 2z$. By the divergence theorem,

$$\iint_{\partial D} F.\vec{n}d\sigma = \iiint_D 2(x+y+z)dV = 2\int\int_{x^2+y^2\le 1} (\int_0^{x+2} (x+y+z)dz)dxdy = \frac{19\pi}{4}$$

Assignment 13 - Figures

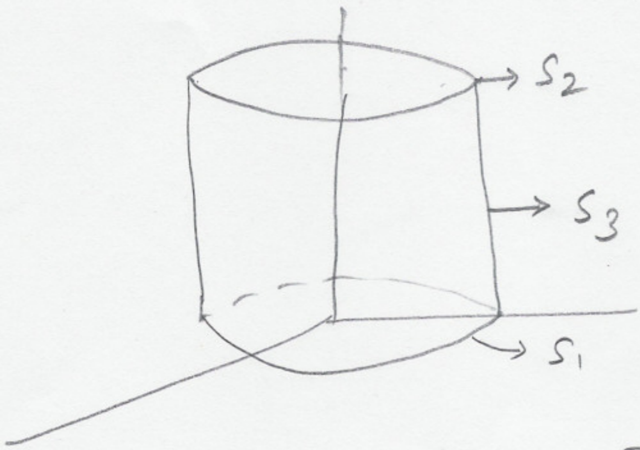
Q 3:





Projection over xy-plane

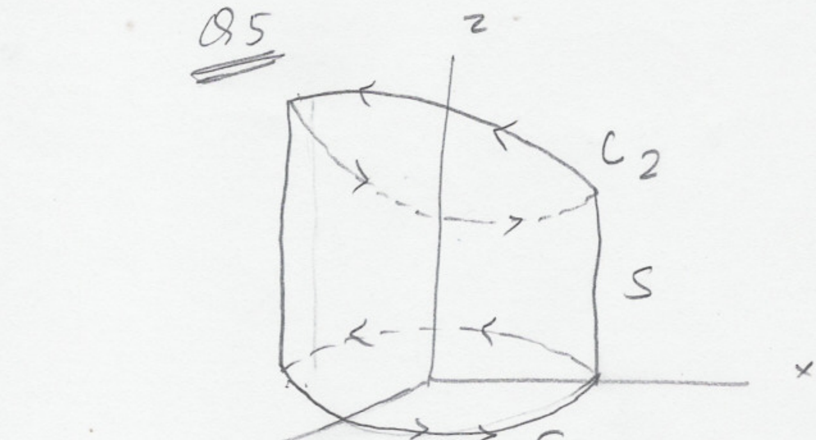
Q 4:



Assignment 14 - Figures

Q 3:

Q 5



b