Assignment 1 : Real Numbers, Sequences

1. **(D)** Find the supremum and infimum of the sets $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$ and $\left\{\frac{m}{|m|+n} : n \in \mathbb{N}, m \in \mathbb{Z}\right\}$.

2. **(D)** Let (x_n) be a sequence of strictly positive real numbers such that $\lim\limits_{n \longrightarrow \infty} \frac{x_{n+1}}{x_n} = \ell$. Then prove the following:

   (a) if $\ell < 1$ then $\lim\limits_{n \longrightarrow \infty} x_n = 0$,

   (b) if $\ell > 1$ then $\lim\limits_{n \longrightarrow \infty} x_n = \infty$

   (c) if $\ell = 1$ then give example of sequences to show that both conclusions can hold.

3. Investigate the convergence of the following sequences:

   (b) **(D)** $x_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}$,

   (d) **(D)** $x_n = \frac{n^s}{(1+p)^n}$ for some $s > 0$ and $p > 0$,

   (e) **(D)** $x_n = \frac{2^n}{n!}$.

5. **(D)** Suppose that $0 < \alpha < 1$ and that (x_n) is a sequence which satisfies one of the following conditions

   (a) $|x_{n+1} - x_n| \leq \alpha^n$, $\qquad n = 1, 2, 3, \ldots$

   (b) $|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|$, $\qquad n = 1, 2, 3, \ldots$.

   Then prove that (x_n) satisfies the Cauchy criterion. *Whenever you use this result, you have to show that the number α that you get, satisfies $0 < \alpha < 1$. The condition $|x_{n+2} - x_{n+1}| \leq |x_{n+1} - x_n|$ does not guarantee the convergence of (x_n). Give examples.*

Assignment 1- Solutions

1. First note that $0 < \frac{m}{m+n} < 1$. We guess that inf $= 0$ because $\frac{1}{1+n}$ is in the set and it approaches 0 when n is very large. Formally to show that 0 is the infimum, we have to show that a number $\alpha > 0$ cannot be a lower bound of the given set. This is true because we can find an n such that $\frac{1}{1+n} < \alpha$ (using the Archimedean property). Similarly we can show that sup $= 1$.

   For the other set inf = -1 and sup = 1.

2. (a) Since $\ell < 1$, we can find an r such that $\ell < r < 1$. As $\lim_{n\longrightarrow\infty} \frac{x_{n+1}}{x_n} = \ell$, there exists n_0 such that $\frac{x_{n+1}}{x_n} < r$, for all $n \geq n_0$.
   Hence, we have

$$0 < x_{n+n_0} < rx_{n+n_0-1} < r^2 x_{n+n_0-2} < \cdots < r^n x_{n_0}.$$

   Since $\lim_{n\longrightarrow\infty} r^n = 0$, (as $0 < r < 1$), by the sandwich theorem, $0 \leq \lim_{n\longrightarrow\infty} x_n \leq 0$. Hence, $x_n \to 0$.

   (b) Since $\ell > 1$, we can find $r \in \mathbb{R}$, such that $1 < r < \ell$. Arguing along the same lines as in (a), we get $n_0 \in \mathbb{N}$, such that $x_{n+1} > rx_n$, $\forall\, n \geq n_0$. Now, $x_{n+n_0} > r^n x_{n_0}$. Since $r > 1$, $\lim_{n\longrightarrow\infty} r^n = \infty$ and therefore $\lim_{n\longrightarrow\infty} x_n = \infty$.

   (c) If $(x_n) = (n)$, then $\lim \frac{x_{n+1}}{x_n} = 1$, but $\lim_{n\longrightarrow\infty} x_n = \infty$.

   If $x_n = \frac{1}{n}$, then $\lim \frac{x_{n+1}}{x_n} = 1$, but $\lim_{n\longrightarrow\infty} x_n = 0$.

   If $x_n = c + \frac{1}{n}$, then $\lim \frac{x_{n+1}}{x_n} = 1$, but $\lim_{n\longrightarrow\infty} x_n = c$.

3.

   (b) $\frac{n.n^2}{n^3+2n} \leq x_n \leq \frac{n.n^2}{n^3+n+1}$. By sandwich theorem, $x_n \to 1$.

   (d) $\frac{x_{n+1}}{x_n} = \frac{(n+1)^s(1+p)^n}{n^s(1+p)^{n+1}} = \frac{1}{1+p}(1+\frac{1}{n})^s \to \frac{1}{p+1} < 1$. Hence, by Problem 2(a), $\lim_{n\longrightarrow\infty} x_n = 0$.

   (e) Consider $\frac{x_{n+1}}{x_n}$ and apply Problem 2(a). Here $x_n \to 0$.

5. (a) Let $n > m$.

Then $|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m|$.

$$|x_n - x_m| \le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$$
$$\le \alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m = \alpha^m[1 + \alpha + \cdots + \alpha^{n-1+m}]$$
$$\le \alpha^m[1 + \alpha + \cdots +]$$
$$= \tfrac{\alpha^m}{1-\alpha} \to 0 \text{ as } m \to \infty.$$

Thus (x_n) satisfies the Cauchy criterion.

(b) Note that $|x_{n+2} - x_{n+1}| \le \alpha|x_{n+1} - x_n| \le \alpha^2|x_n - x_{n-1}| \le \cdots \le \alpha^n|x_2 - x_1|$.

For $n > m$,

$$|x_n - x_m| \le (\alpha^{n-2} + \alpha^{n-3} + \cdots + \alpha^{m-1})|x_2 - x_1|$$
$$\le \tfrac{\alpha^m}{1-\alpha}|x_2 - x_1| \to 0 \text{ as } m \to \infty.$$

Thus (x_n) satisfies the Cauchy criterion.

Examples:

(i) $x_n = n$. Here, $|x_{n+2} - x_{n+1}| = 1 = |x_{n+1} - x_n|$.

(ii) $x_n = \sqrt{n}$. Here,

$|x_{n+2} - x_{n+1}| = |\sqrt{n+2} - \sqrt{n+1}| = \frac{1}{\sqrt{n+2}+\sqrt{n+1}} \le \frac{1}{\sqrt{n+1}+\sqrt{n}} = |x_{n+1} - x_n|$.