Assignment 2 : Continuity, Intermediate Value Property

1. (D) Let \( f : \mathbb{R} \to \mathbb{R} \) be such that for every \( x, y \in \mathbb{R} \), \( |f(x) - f(y)| \leq |x - y| \). Show that \( f \) is continuous.

3. (D) Let \( f : (-1, 1) \to \mathbb{R} \) be a continuous function such that in every neighborhood of 0, there exists a point where \( f \) takes the value 0. Show that \( f(0) = 0 \).

5. (D) Let \( f : \mathbb{R} \to \mathbb{R} \) satisfy \( f(x + y) = f(x) + f(y) \) for all \( x, y \in \mathbb{R} \). If \( f \) is continuous at 0, show that \( f \) is continuous at every point \( c \in \mathbb{R} \).

8. (D) Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Show that the range \( \{ f(x) : x \in [a, b] \} \) is a closed and bounded interval.

10. (D) Show that a polynomial of odd degree has at least one real root.
Assignment 2 - Solutions

1. Let \( x_0 \in \mathbb{R} \) and \( x_n \to x_0 \). Since \( |f(x_n) - f(x_0)| \leq |x_n - x_0| \), \( f(x_n) \to f(x_0) \). Therefore \( f \) is continuous at \( x_0 \).

3. There exists \( x_n \in (-\frac{1}{n}, \frac{1}{n}) \) such that \( f(x_n) = 0 \). Since \( f \) is continuous at 0 and \( x_n \to 0 \), we have \( f(x_n) \to f(0) \). Therefore, \( f(0) = 0 \).

5. First note that \( f(0) = 0, f(-x) = -f(x) \) and \( f(x - y) = f(x) - f(y) \). Let \( x_0 \in \mathbb{R} \) and \( x_n \to x_0 \). Then \( f(x_n) - f(x_0) = f(x_n - x_0) \to f(0) = 0 \) as \( f \) is continuous at 0 and \( x_n - x_0 \to 0 \).

8. Let \( x_0, y_0 \in [a, b] \) such that \( f(x_0) = m = \inf f \) and \( f(y_0) = M = \sup f \). Suppose \( x_0 < y_0 \). By IMP, for every \( \alpha \in [m, M] \) there exists \( x \in [x_0, y_0] \) such that \( f(x) = \alpha \). Hence \( f([a, b]) = [m, M] \).

10. Let \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \), \( a_n \neq 0 \) and \( n \) be odd. Then \( p(x) = x^n (a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}) \). If \( a_n > 0 \), then \( p(x) \to \infty \) as \( x \to \infty \) and \( p(x) \to -\infty \) as \( x \to -\infty \). Thus by the intermediate value property, there exists \( x_0 \) such that \( p(x_0) = 0 \). Similar argument for \( a_n < 0 \).