Assignment 2 : Continuity, Intermediate Value Property

1. **(D)** Let $f : \mathbb{R} \to \mathbb{R}$ be such that for every $x, y \in \mathbb{R}$, $| f(x) - f(y) | \leq | x - y |$. Show that f is continuous.

3. **(D)** Let $f : (-1, 1) \to \mathbb{R}$ be a continuous function such that in every neighborhood of 0, there exists a point where f takes the value 0. Show that $f(0) = 0$.

5. **(D)** Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every point $c \in \mathbb{R}$.

8. **(D)** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that the range $\{f(x) : x \in [a, b]\}$ is a closed and bounded interval.

10. **(D)** Show that a polynomial of odd degree has at least one real root.

Assignment 2 - Solutions

1. Let $x_0 \in \mathbb{R}$ and $x_n \to x_0$. Since $\mid f(x_n) - f(x_0) \mid \leq \mid x_n - x_0 \mid, f(x_n) \to f(x_0)$. Therefore f is continuous at x_0.

3. There exists $x_n \in (-\frac{1}{n}, \frac{1}{n})$ such that $f(x_n) = 0$. Since f is continuous at 0 and $x_n \to 0$, we have $f(x_n) \to f(0)$. Therefore, $f(0) = 0$.

5. First note that $f(0) = 0, f(-x) = -f(x)$ and $f(x-y) = f(x) - f(y)$. Let $x_0 \in \mathbb{R}$ and $x_n \to x_0$. Then $f(x_n) - f(x_0) = f(x_n - x_0) \to f(0) = 0$ as f is continuous at 0 and $x_n - x_0 \to 0$.

8. Let $x_0, y_0 \in [a, b]$ such that $f(x_0) = m = inff$ and $f(y_0) = M = supf$. Suppose $x_0 < y_0$. By IMP, for every $\alpha \in [m, M]$ there exists $x \in [x_0, y_0]$ such that $f(x) = \alpha$. Hence $f([a, b]) = [m, M]$.

10. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$ and n be odd. Then $p(x) = x^n(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n})$. If $a_n > 0$, then $p(x) \to \infty$ as $x \to \infty$ and $p(x) \to -\infty$ as $x \to -\infty$. Thus by the intermediate value property, there exists x_0 such that $p(x_0) = 0$. Similar argument for $a_n < 0$.