Assignment 3 : Derivatives, Maxima and Minima, Rolle's Theorem

3. **(D)** Show that the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at all $x \in \mathbb{R}$. Also show that the function $f'(x)$ is not continuous at $x = 0$. Thus, a function that is differentiable at every point of $\mathbb{R}$ need not have a continuous derivative $f'(x)$.

5. **(D)** Let $f(0) = 0$ and $f'(0) = 1$. For a positive integer$k$, show that

$$\lim_{x \to 0} \frac{1}{x} \left\{ f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + ... + f(\frac{x}{k}) \right\} = 1 + \frac{1}{2} + ... + \frac{1}{k}$$

7. **(D)** Prove that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one real root.

10. **(D)** Let f and g be functions, continuous on $[a,\ b]$, differentiable on $(a,\ b)$ and let $f(a) = f(b) = 0$. Prove that there is a point $c \in (a,\ b)$ such that $g'(c)f(c) + f'(c) = 0$.

Assignment 3 - Solutions

3. Using the sandwich theorem, we can see $\lim\limits_{h\to 0} h\sin\frac{1}{h} = 0$. Therefore, f is differentiable at 0 and $f'(0) = 0$.

   Now,
$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   Since $\lim\limits_{h\to 0}\cos\frac{1}{h}$ does not exist, $f'(x)$ is not continuous at 0.

5. $\lim\limits_{x\to 0}\frac{1}{x}(f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + \cdots + f(\frac{x}{k})) =$
$\lim\limits_{x\to 0}(\frac{f(x)-f(0)}{x} + \frac{1}{2}\frac{f(\frac{x}{2})-f(0)}{\frac{x}{2}} + \cdots + \frac{1}{k}\frac{f(\frac{x}{k})-f(0)}{\frac{x}{k}}) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

7. Let $f(x) = x^{13} + 7x^3 - 5$. Here, $f(x) < 0 \ \forall\ x \leq 0$, $f(0) = -5$ and $f(1) = 3$. By the intermediate value property, there exists $c \in (0,1)$, such that $f(c) = 0$. So, f has at least one real root.

   If f has more than one real roots, (from above) they must all be positive. But, $f'(x) = x^2(13x^{10} + 21) \neq 0$ unless $x = 0$. Since $f'(x)$ has no positive root, f has atmost one real root.

10. Define $h(x) = f(x)e^{g(x)}$. Here, $h(x)$ is continuous in $[a,b]$ and differentiable in (a,b). Since $h(a) = h(b) = 0$, by Rolle's theorem, $\exists\ c \in (a,b)$ such that $h'(c) = 0$.

    Since $h'(x) = [f'(x) + g'(x)f(x)]e^{g(x)}$ and $e^{\alpha} \neq 0$ for any $\alpha \in \mathbb{R}$, we see that $f'(c) + g'(c)f(c) = 0$.