Assignment 4 : Mean Value Theorem, Taylor's Theorem, Curve Sketching

2. **(D)** Let $a > 0$ and $f : [-a, a] \to \mathbb{R}$ be continuous. Suppose $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, then show that $f(x) = x$ for every $x \in (-a, a)$.

4. Using Cauchy Mean Value Theorem, show that

   (a) **(D)** $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$.

5. **(D)** Let f be continuous on $[a, b]$, $a > 0$ and differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$\frac{bf(a) - af(b)}{b - a} = f(c) - cf'(c).$$

9. **(D)** Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that

$$x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k}x^{2k} \; < \; \log(1 + x) \; < \; x - \frac{1}{2}x^2 + \cdots + \frac{1}{2k+1}x^{2k+1}.$$

Assignment 4 - Solutions

2. Let $g(x) = f(x) - x$ on $[-a, a]$. Note that $g'(x) \leq 0$ on (-a,a). Therefore, g is decreasing. Since $g(a) = g(-a) = 0$, we have $g = 0$.

   This problem can also be solved by applying MVT for g on $[-a, x]$ and $[x, a]$.

4. (a) Apply CMVT to $f(x) = 1 - \cos x$ and $g(x) = \frac{x^2}{2}$. We get $\frac{1-\cos x}{x^2/2} = \frac{\sin c}{c} < 1$ for some c between 0 and x.

5. Apply CMVT to $\frac{f(x)}{x}$ and $\frac{1}{x}$.

9. By Taylor's theorem, $\exists\, c \in (0, x)$ s.t. $\log(1+x) = x - \frac{1}{2}x^2 + \ldots + \frac{(-1)^{n-1}}{n}x^n + \frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}$. Note that, for any $x > 0$, $\frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}} > 0$ if $n = 2k$ and < 0 if $n = 2k + 1$.