Assignment 5 : Series, Power Series, Taylor Series

1. **(D)** Let $a_n \geq 0$. Then show that both the series $\sum\limits_{n\geq 1} a_n$ and $\sum\limits_{n\geq 1} \frac{a_n}{a_n+1}$ converge or diverge together.

3. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n\geq 1} a_n$ where a_n equals:

   (a)**(D)** $1-n\sin\frac{1}{n}$ (b)**(D)** $\frac{1}{n}\log(1+\frac{1}{n})$

5. **(D)** Let $\{a_n\}$ be a decreasing sequence, $a_n \geq 0$ and $\lim\limits_{n\longrightarrow\infty} a_n = 0$. For each $n \in \mathbb{N}$, let $b_n = \frac{a_1+a_2+\cdots+a_n}{n}$. Show that $\sum\limits_{n>1}(-1)^n b_n$ converges.

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Assignment 5- Solutions

(1) Suppose $\sum_{n\geq 1} a_n$ converges. Since $0 \leq \frac{a_n}{1+a_n} \leq a_n$ by comparison test $\sum_{n\geq 1} \frac{a_n}{1+a_n}$ converges.

Suppose $\sum_{n\geq 1} \frac{a_n}{1+a_n}$ converges. By the necessary condition, $\frac{a_n}{1+a_n} \to 0$. Hence $a_n \to 0$ and therefore $1 \leq 1 + a_n < 2$ eventually. Hence $0 \leq \frac{1}{2}a_n \leq \frac{a_n}{1+a_n}$. Apply the comparison test.

(3) (a) Use Limit Comparison Test (LCT) with $\frac{1}{n^2}$. Since $1 - n\sin\frac{1}{n} \leq \frac{1}{3!n^2} < \frac{1}{n^2}$, one can also use comparison test. *(We will tell in the class, how to guess "b_n" and apply the LCT. So, the students might feel that the LCT is easier to apply compared to the comparison test).*

(b) Use LCT or comparison test with $\frac{1}{n^2}$.

(5) $b_{n+1} - b_n = \frac{1}{n+1}(a_1 + a_2 + \dots a_{n+1}) - \frac{1}{n}(a_1 + \cdots + a_n) = \frac{a_{n+1}}{n+1} - \frac{(a_1+\cdots+a_n)}{n(n+1)}$. Since (a_n) is decreasing, $a_1 + \dots a_n \geq na_n$. Therefore, $b_{n+1} - b_n \leq \frac{a_{n+1}-a_n}{n+1} \leq 0$. Therefore, (b_n) is decreasing.

We now need to show that $b_n \to 0$. For a given $\epsilon > 0$, since $a_n \to 0$, there exists n_0 such that $a_n < \frac{\epsilon}{2}$, $\forall$, $n \geq n_0$.

Therefore, $|\frac{a_1+\cdots+a_n}{n}| = |\frac{a_1+\cdots+a_{n_0}}{n} + \frac{a_{n_0+1}+\cdots+a_n}{n}| \leq |\frac{a_1+\cdots+a_{n_0}}{n}| + \frac{n-n_0}{n}\frac{\epsilon}{2}$. Choose $N \geq n_0$ large enough so that $\frac{a_1+\cdots+a_{n_0}}{N} < \frac{\epsilon}{2}$. Then, for all $n \geq N$, $\frac{a_1+\dots a_n}{n} < \epsilon$. Hence, $b_n \to 0$. Use the Leibniz test for convergence.